Comparison of Electromagnetic Antenna Chu Limit and Q of Gravitational Radiators

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***Abstract*—Direct observations by the Laser Interferometer Gravitational-Wave Observatory (LIGO) since 2015 have corroborated general relativity predictions of gravitational-wave phenomena. Following this, an analytic expression has been found for the Q of gravitational quadrupole radiators, where Q was shown to be a function of the physical size of the gravitational-wave source. This new result is similar to the electromagnetic Chu limit, where the Q of electrically-small antennas is limited by the physical size of an antenna. In this paper, initial observations and comparisons are made between gravitational Q and electromagnetic Q over a range of physical parameters. The results illustrate a number of similarities and differences between gravitational Q and electromagnetic Q.**

***Index Terms*—gravitational waves, waves, Antenna theory, Q measurement**

I. INTRODUCTION

Since the first observation of gravitational-wave event GW150914 in September of 2015, data collected by the Laser Interferometer Gravitational-Wave Observatory (LIGO) has provided direct confirmation of the existence of gravitational waves [1], [2]. Within the past year, a new analytic expression for the Q (quality factor) of gravitational-wave sources was found, where the Q of gravitational quadrupoles was shown to depend on the physical size of the source [3]. This new result for gravitational wave phenomena was inspired by the Chu limit (or Wheeler-Chu limit) for gravitationally-small antennas [4]–[6]. Given these new theoretical results, this paper provides initial comparisons between the theoretical Q of gravitationally-small gravitational quadrupole radiation sources and the more than 50-year-old Chu-limit for the theoretical Q of electrically-small antennas.

In the following section, the theory of gravitational quadrupole Q is reviewed along with a brief review of the Chu limit and theoretical Q of electrically small antennas. Important differences between gravitational Q and electromagnetic Q are noted. The subsequent section provides initial comparisons of results for illustrative examples of gravitational quadrupole Q and electrically-small antenna Q plotted over a range of parameter values. The results illustrate several notable differences between gravitational Q and electromagnetic Q.

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II. THEORY

Before reviewing the Chu limit for the Q of electrically-small antennas, we first review the Q of gravitationally-small ($size \ll \lambda$) gravitational quadrupoles. In particular, we consider gravitational quadrupoles formed by two masses m_1 and m_2 in circular orbit about a barycenter C as illustrated in Fig. 1. As the masses orbit each other, gravitational waves are emitted with luminosity $\mathcal{L} = 32(m_1 + m_2)^5 \nu^2 G^4/(5d_s^5 c^5)$ for two orbiting masses m_1 and m_2 in kg, with $\nu = m_1 m_2/(m_1+m_2)^2$, and where G is the gravitational constant 6.7×10^{-11} N·(m/kg)^2 [2], [7], [8].

For the scenario of Fig. 1, the Q of a gravitational quadrupole source of gravitational waves has been shown to be [3]

$$Q_g = \frac{20 m_1^7 G}{c^2 m_2 (m_1 + m_2)^5} \left[\frac{(k_g a_g)^{-7}}{2 a_{min}} - \frac{(k_g a_g)^{-5} c^2 (m_1 + m_2)^2}{8 m_1^3 G} \right] , \quad (1)$$

where a_g of Fig. 1 is the larger orbital radius in meters, c is the speed of light in vacuum, $k_g = 2\pi f_g / c$ is the gravitational wavenumber, f_g is the gravitational wave frequency in Hz, and a_{min} is the final radius of the larger orbit around the barycenter at coalescence [3]. Lastly, note that orbital radius a_g corresponds to the radius of a sphere that would enclose the physical dimensions of the quadrupole comprised of the orbiting masses.

By comparison, the electromagnetic Chu-limit Q of an electrically-small antenna is [5]

$$Q_{em} = \frac{1}{k_{em}\, a_{em}} + \frac{1}{(k_{em}\, a_{em})^3} \quad \text{for } k_{em}\, a_{em} \ll 1 , \quad (2)$$

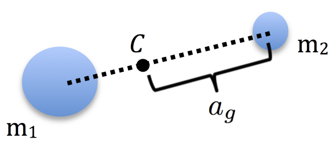

Fig. 1. Gravitational quadrupole consisting of two masses m_1 and m_2 in orbit around barycenter C, with larger orbital radius being a_g.

where $k_{em} = 2\pi f_{em}/c$ is the electromagnetic wavenumber, f_{em} is the electromagnetic wave frequency in Hz, and a_{em} is the radius of a sphere that would enclose the antenna. Thus, the foregoing size parameter $k_g a_g$ in (1) of a sphere that would enclose the gravitational radiation source is analogous to the size parameter $k_{em} a_{em}$ in (2) for an electrically-small antenna.

By comparing (1) with (2), we may observe several key differences between the theoretical Q of gravitational and electromagnetic radiation sources. First, the gravitational Q_g varies as $(k_g a_g)^{-7}$ for $k_g a_g \ll 1$, whereas the electromagnetic Q_{em} can be seen to vary as $(k_{em} a_{em})^{-3}$ for $k_{em} a_{em} \ll 1$. Second, the gravitational Q_g also depends on additional physical parameters of the system, including m_1, m_2, and a_{min}, whereas Q_{em} for electrically-small antennas only depends on $k_{em} a_{em}$. In addition, an example below is used to show that gravitational Q_g also depends on the ratio of the two masses m_1 and m_2 that comprise the gravitational quadrupole.

III. Illustrative Examples and Comparison

To illustrate the differences between theoretical gravitational quadrupole Q_g and theoretical Q_{em} of electrically small antennas, several examples are plotted in Fig. 2. In this plot, the horizontal axis represents ka for the cases of $k_{em} a_{em}$ or $k_g a_g$, as appropriate. The lower dashed curve shows theoretical Q_{em} of electrically small antennas from (2), while the two upper curves illustrate two different cases of gravitational Q_g from (1).

As expected, the lower dashed curve showing theoretical Q_{em} of electrically-small antennas increases by a factor of 10^3 as $k_{em} a_{em}$ decreases from 0.1 to 0.01. In contrast, the middle solid curve shows that gravitational Q_g increases by a factor of $\approx 10^7$ as $k_g a_g$ decreases from 0.1 to 0.01, where equal masses of $m_1 = m_2 = 2.9 \times 10^{30}$ kg were used (similar to the total binary neutron star mass in GW170817), with $a_{min} \approx$ 29 km [7], [9].

Lastly, gravitational Q_g not only depends on total mass $m_1 + m_2$, but also depends on the distribution of the mass between the two orbiting objects. This is illustrated in the upper dot-dashed curve in Fig. 2 using the same total mass as the solid curve, but with $m_1 = 3m_2$. As before, the dot-dashed curve shows that Q_g increases by a factor of $\approx 10^7$ as $k_g a_g$ decreases from 0.1 to 0.01, since the dashed curve is essentially parallel to the solid curve. Despite the equal total mass for both cases, at $ka = 0.1$ the value of $Q_g \approx 7.6 \times 10^6$ with $m_1 = 3m_2$ is approximately 38 times larger than the value $Q_g \approx 2 \times 10^5$ with $m_1 = m_2$.

IV. Summary

The theoretical Q_g for gravitationally-small gravitational radiation sources is compared to the theoretical Q_{em} for electrically-small antennas. Most significantly, the Q of small gravitational and electromagnetic sources both strongly depend on physical size of the radiation source. However, the power laws differ significantly, with gravitational Q_g being proportional to $(k_g a_g)^{-7}$, and with electrically-small antenna Q_{em} being proportional to $(k_{em} a_{em})^{-3}$. In addition, gravitational

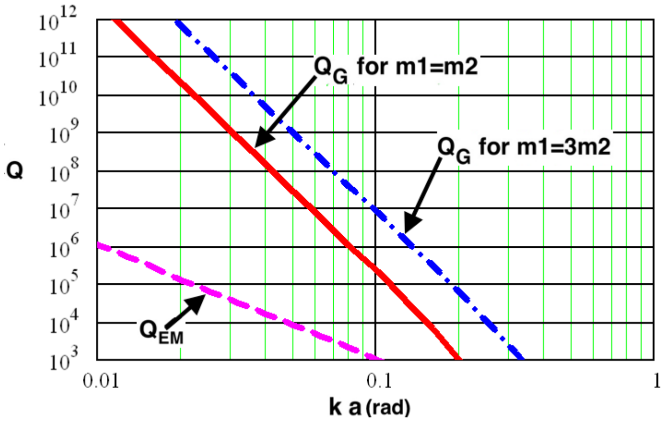

Fig. 2. Q as a function of size parameter ka. Lower dashed curve is theoretical Q_{em} as a function of $k_{em} a_{em}$ for electrically-small antennas. Middle solid curve is theoretical gravitational Q_g as a function of $k_g a_g$ for equal masses of $m_1 = m_2 = 2.9 \times 10^{30}$ kg. Upper dot-dashed curve is theoretical gravitational Q_g as a function of $k_g a_g$ for $m_1 = 3m_2$, and having the same total mass as for the solid curve.

Q_g is shown to be affected by the ratio m_2/m_1 of the two masses in orbit. Plots of several examples illustrate the differences between gravitational and electromagnetic Q. Lastly, it remains to be seen whether the additional degrees of freedom, such as dependence of Q_g on the ratio m_2/m_1, can be used to provide insights for improving the design of gravitational detectors or sources.

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