Observed Q and Gravitationally-Small Antenna Behavior of a Binary Black Hole Radiator

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***Abstract*—Recent theoretical advances have provided an analytical expression for the Q of gravitationally-small sources of gravitational radiation, along lines similar to the Wheeler-Chu limit for electrically-small antennas. This paper presents the first published results using this new theory to analyze the Q of a black-hole inspiral, using observed transient gravitational wave data from the GW170608 black-hole merger. Despite the astronomical scale of a radiation source comprised of two black holes having upwards of 7 solar masses each, the GW170608 binary black hole is shown to be a gravitationally-small high-Q radiator at the very low frequencies of the gravitational waves.**

I. INTRODUCTION

The first direct observation of gravitational waves was made on September 14, 2015 by LIGO (Laser Interferometer Gravitational-Wave Observatory) from a merging binary system of black holes producing gravitational-wave event GW150914 [1]. Early investigations in [2] suggested that neutron-star gravitational-wave sources were gravitationally small (size $\ll \lambda/\pi$) and should exhibit a gravitational Q, similar to the electromagnetic Q of electrically-small antennas [3]. More recent theoretical results in [4] provided an analytic expression for the Q of gravitational sources, and provided the observed Q for a binary neutron-star gravitational-wave source. In this paper, we show results for the observed Q of the GW170608 binary black-hole gravitational-wave source [5]. Despite the considerably larger masses of the black holes, the GW170608 binary black-hole source is shown below to be gravitationally small and to have high Q.

The following results are the first published data that presents a binary black-hole inspiral as a gravitationally-small source for gravitational radiation. Earlier results were for a binary neutron star, and given the significantly larger masses of black holes, it was unclear whether binary black holes would be gravitationally-small radiators. In the following, the changing size of the radiation source during inspiral provides the first observed Q for a binary black hole over a range of gravitationally-small dimensions. Beyond these fundamental results, the gravitationally-small high Q characteristics raise the question of what antenna engineering techniques may be applied to improve gravitational-wave receivers.

II. THEORY OF GRAVITATIONAL Q

Recently, in [4] we derived an analytic expression for the Q of a gravitational-radiation source, along the lines of the Chu limit of electromagnetic radiation sources [3]. In the case of electromagnetic antennas, the Chu limit is $Q \approx 1/(ka)^3$ for $k\,a \ll 1$, where $k = 2\pi f_0/c = 2\pi f_0/(3 \times 10^8)$, and a is the radius of the sphere enclosing the antenna [3]. In the case of gravitational waves, we have shown the Q of a gravitational source to be [4]

$$Q = \frac{20 m_1^7 G}{m_2 c^2 (m_1+m_2)^5} \left(\frac{(k\,a_s)^{-7}}{2a_{min}} - \frac{(k\,a_s)^{-5} c^2 (m_1+m_2)^2}{8 m_1^3 G} \right), \quad (1)$$

where the gravitational constant $G = 6.7 \times 10^{-11}$ N·(m/kg)^2. For the case of GW170608 [5], the masses are $m_1 = 2.4 \times 10^{31}$ kg and $m_2 = 1.4 \times 10^{31}$ kg, and a_{min} is the final radius of the orbit around the barycenter at coalescence. From prior theoretical results in [4], a_{min} at coalescence can be found from the gravitational-wave frequency at coalescence using

$$a_s = \left(\frac{m_1^3 G}{\omega_{orb}^2 (m_1+m_2)^2} \right)^{1/3} , \quad (2)$$

where a_s is radius of the orbit around the barycenter, $\omega_{orb} = 2\pi f_{orb} = \pi f_{gw}$ is the orbital frequency in rad/s, and f_{gw} is the gravitational-wave frequency in Hz. Then, for the 531.5 Hz average peak GW strain frequency from [6] that occurs near coalescence [7], the value of a_{min} in (1) is calculated from (2) to be $a_{min} \approx 60$ km.

From prior results in [4], the size parameter $k\,a_s$ in (1) is

$$k\,a_s \;=\; \frac{2m_1}{c} \left(\frac{\omega_{orb}\, G}{(m_1+m_2)^2} \right)^{1/3} . \quad (3)$$

Lastly, f_{gw} is estimated from a curve-fit to the well-known form of the frequency chirp of a binary inspiral [4]

$$\frac{1}{f_{gw}} = \frac{8\pi}{125^{1/8}} \left(\frac{G^{5/3} m_1 m_2}{c^5\,(m_1+m_2)^{1/3}}\, t' \right)^{3/8} , \quad (4)$$

where $t' = -t$ is the time before coalescence. Taking the natural logarithm,

$$\ln(f_{gw}) = 3.72 - 3\ln(t')/8 , \quad (5)$$

for the foregoing masses m_1 and m_2.

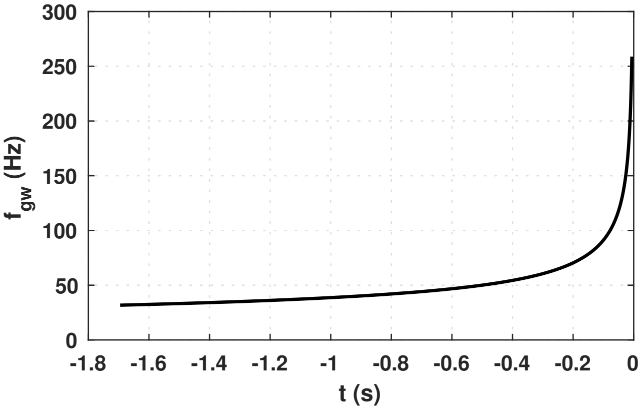

Fig. 1. Observed gravitational wave frequency f_{gw} in Hz for binary black-hole inspiral GW170608 as a function of time before coalescence. Solid black curve is observed gravitational wave frequency f_{gw} estimated from the time-frequency map in [5]. Coalescence is at t = 0.

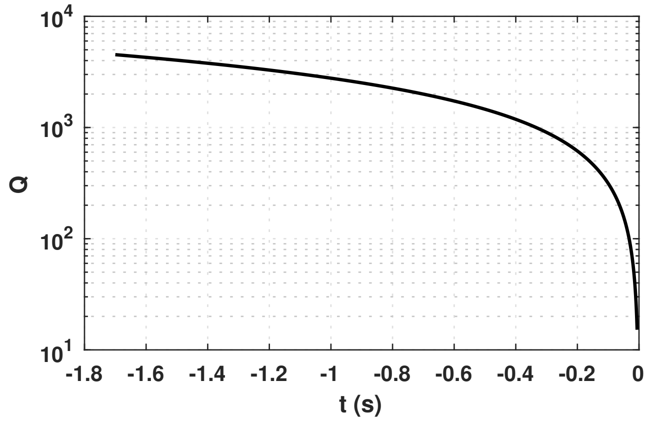

Fig. 3. Observed Q for the binary black-hole inspiral GW170608 as a function of time before coalescence. Solid black curve is the observed Q computed from the value of $k\,a_s$ in Fig. 2. Coalescence is at t = 0.

III. OBSERVED GRAVITATIONAL Q FOR GW170608

The gravitational wave GW170608 observed by LIGO in June 2017 was caused by the inspiral and merger of a binary black hole [5]. Fig. 1 shows observed gravitational wave frequency f_{gw} in Hz, as estimated from the LIGO-Hanford time-frequency map in [5]. The gravitational-wave frequency of Fig. 1 was estimated with a log-log fit to (4) using the time-frequency map published in [5], resulting in a curve-fit of $\ln(f_{gw}) = 3.655 - 0.3725\ln(t')$, for f_{gw} from approximately 32 Hz to 259 Hz. This result is in good agreement with the theoretical relation of (5), with 1.75% error for the constant term, and 0.67% error for the slope term.

Fig. 2 shows the observed gravitational wave size parameter $k\,a_s$ in radians as determined by (3), using the gravitational-wave frequency f_{gw} shown in Fig. 1. Importantly, the black-hole radiator remains gravitationally small during inspiral with $k\,a_s$ changing from $k\,a_s \approx 0.27$ to $k\,a_s \approx 0.54$. As the frequency increases in Fig. 1, $k\,a_s$ in Fig. 2 also increases, in accordance with $k\,a_s$ being proportional to $\omega_{orb}^{1/3} = (\pi f_{gw})^{1/3}$ from (3). During inspiral, the increase in $k\,a_s$ while orbital radius a_s is decreasing seems counterintuitive, but this is caused by the frequency f_{gw} increasing faster than the orbital radius decreases.

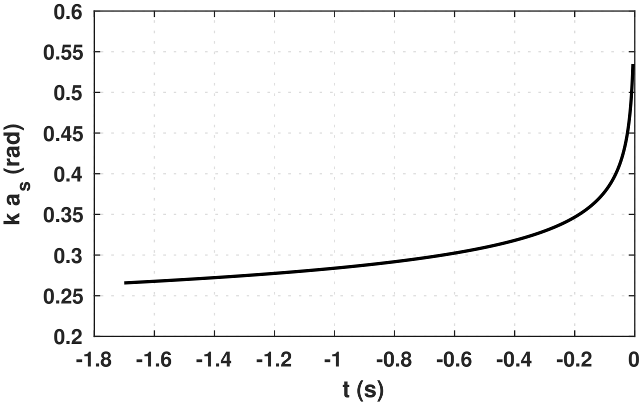

Fig. 2. Observed $k\,a_s$ in rad for binary black-hole inspiral GW170608 as a function of time before coalescence. Solid black curve is observed gravitational wave size parameter $k\,a_s$ computed from the observed gravitational wave frequency f_{gw} in Fig. 1. Coalescence is at t = 0.

Fig. 3 shows the observed Q of GW170608 computed from (1) using the observed value of $k\,a_s$ from Fig. 2. Here, Q decreases as both f_{gw} and $k\,a_s$ increase. The Q of the gravitational-wave source varies from a maximum of $Q \approx 4520$ at $t \approx -1.7$ s to $Q \approx 15$ near coalescence. It is an open question whether antenna engineering concepts such as the Chu limit and electrically-small antennas can be applied to improve the design of gravitational-wave detectors and to increase understanding of gravitational waves. Nevertheless, the observed Q in Fig. 3 would seem to suggest there may yet be untapped opportunity in applying electrically-small antenna engineering methods to gravitational-wave problems.

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