Prospective Gamma-Ray Propagation Models and Comparisons to Metamaterial Models

Thomas P. Weldon, Ryan S. Adams, Kasra Daneshvar, and Raghu K. Mulagada

Department of Electrical and Computer Engineering
University of North Carolina at Charlotte
Charlotte, NC 28223, USA
tpweldon@uncc.edu

***Abstract*—Astronomical observations of gamma-ray bursts can exhibit dispersive behavior where high-energy gamma rays arrive significantly later than low-energy photons. Although the underlying mechanisms for the dispersion are not fully understood, a polynomial model has been proposed for modeling the apparent frequency-dependent photon velocity. The present work considers the impact of this dispersion model on the Helmholtz equation. The result is an extended form of the Helmholtz equation, where additional terms are used to model observed dispersion. The proposed set of equations closely resembles one type of metamaterial model, and therefore exhibits similar behavior. Comparisons are drawn between the gamma-ray models and similar metamaterial models that exhibit right-handed behavior at low frequencies and left-handed behavior at high frequencies. Finally, the overall approach provides a flexible modeling framework that can be adapted as new gamma-ray data become available.**

I. Introduction

The Fermi Gamma-ray Space Telescope has enhanced the ability of scientists to measure dispersion of high-energy gamma rays [1]. While the underlying mechanism of the dispersion remains uncertain, a number of investigators have suggested a quadratic polynomial model for the frequency-dependent velocity of high-energy photons [2]–[5]. Although a linear dispersion model appears unlikely based on data from gamma-ray burst GRB 090510 in May of 2009, quadratic dispersion models have not yet been disproven [6]. In this gamma-ray burst, photons at 31 GeV arrived perhaps as much as 859 ms later than low energy photons.

Since linear dispersion models appear unlikely for gamma-ray bursts, the remainder of the paper focuses on developing dispersive equations from quadratic dispersion models and the Helmholtz equation. The following development addresses the formulation of dispersive propagation models based directly on the measured dispersion data of gamma-ray bursts. This empirical approach avoids the need for theoretical details of the physical mechanisms of the dispersion and avoids reliance on controversial theories.

In addition, the proposed dispersion models are observed to have the same form as certain types of metamaterial models [7]. Having the same form of equations as these metamaterials, the proposed gamma-ray dispersion models then exhibit right-handed behavior at low frequency, left-handed behavior at high frequency, and a forbidden frequency band, or stop band. Therefore, these corresponding metamaterial models offer additional insight to the behavior of the proposed gamma-ray models.

In the next section, the proposed gamma-ray propagation models are first developed, based on empirical data from recent gamma-ray burst data. Then, a similar metamaterial model is presented in the subsequent section. Further insight to the behavior of the proposed gamma-ray models is drawn from these metamaterial models.

II. Proposed Gamma-Ray Propagation Models

In previous investigations of dispersion in gamma-ray bursts, the velocity of light has been modeled as a power series expansion for high photon energies [2]–[5]. In these models, the photon velocity u is typically expressed as a function of photon energy $\mathcal{E}$, or of angular frequency ω, up to second order [2]:

$$\begin{aligned} u &= c\left(1-\xi\frac{\mathcal{E}}{\mathcal{E}_p}-\zeta\frac{\mathcal{E}^2}{\mathcal{E}_p^2}\right) \\ &= c\left(1-a_1\omega-a_2\omega^2\right), \end{aligned} \tag{1}$$

where $\mathcal{E}=h\omega/(2\pi)$ is photon energy in eV, ω is photon frequency in rad/s, Planck's constant is $h=4.14\times10^{-15}$ eV·s, $\mathcal{E}_p=1.22\times10^{28}$ eV is the Planck energy, $c=3.0\times10^8$ m/s, and ξ, ζ, $a_1=\xi h/(2\pi\mathcal{E}_p)$, and $a_2=\zeta h^2/(2\pi\mathcal{E}_p)^2$ are free parameters [2].

However, recent measured data from gamma-ray burst GRB 090510 suggest that the linear term a_1 may be zero [6], resulting in:

$$u=c\left(1-a_2\omega^2\right). \tag{2}$$

Next, the foregoing dispersion relation can be substituted into the Helmholtz equation. Because u approaches c at low frequency in (2), the resulting Helmholtz equation will remain consistent with the classical free-space Helmholtz equation at low frequency. To begin, consider the Helmholtz equation in a vacuum for low energy photons [8]:

$$\nabla^2 \mathbf{E} = \frac{-\omega^2}{c^2} \mathbf{E} = -\omega^2 \mu_\circ \epsilon_\circ \mathbf{E}, \tag{3}$$

and the corresponding free-space form of Maxwell's equations:

$$\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu_\circ \mathbf{H} \\ \nabla \times \mathbf{H} &= j\omega\epsilon_\circ \mathbf{E}, \end{aligned} \tag{4}$$

where the vacuum permittivity is $\epsilon_\circ = 8.85 \times 10^{-12}$ F/m and the vacuum permeability is $\mu_\circ = 1.26 \times 10^{-6}$ H/m. When the dispersive velocity u from (2) is substituted for the constant velocity c in (3), the dispersive form of the Helmholtz equation becomes:

$$\nabla^2 \mathbf{E} = -\frac{\mu_\circ}{(1 - a_2\omega^2)} \frac{\epsilon_\circ}{(1 - a_2\omega^2)} \omega^2 \mathbf{E}, \tag{5}$$

where the denominator terms below ϵ_0 and μ_0 were equally distributed on the right side of (5). The plane-wave solution of (5) is $\mathbf{E} = \mathbf{E}_\circ e^{-jkz} e^{j\omega t} = \mathbf{E}_\circ e^{-j\omega z/u} e^{j\omega t}$, where the phase velocity is $u = c(1 - a_2\omega^2)$, and the wavenumber is $k = \omega/u$. Comparison of (5) with (3) further suggests that (4) should then become:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{j\omega\mu_\circ}{(1 - a_2\omega^2)} \mathbf{H} \\ \nabla \times \mathbf{H} &= \frac{j\omega\epsilon_\circ}{(1 - a_2\omega^2)} \mathbf{E}, \end{aligned} \tag{6}$$

or

$$\begin{aligned} \nabla \times \mathbf{E} - a_2\omega^2 (\nabla \times \mathbf{E}) &= -j\omega\mu_\circ \mathbf{H} \\ \nabla \times \mathbf{H} - a_2\omega^2 (\nabla \times \mathbf{H}) &= j\omega\epsilon_\circ \mathbf{E}, \end{aligned} \tag{7}$$

The result in (7) is the proposed form of dispersive Helmholtz equations. This result follows from the quadratic dispersive velocity model in (2) being substituted into the Helmholtz equation in (3). The dispersion parameter a_2 in (7) can be computed from gamma-ray measurements where $\mathcal{E}_p/(\zeta^{1/2}) \approx 5 \times 10^{18}$ eV [2]. Solving for a_2 then gives $a_2 = \zeta h^2/(2\pi\mathcal{E}_p)^2 \approx 1.74 \times 10^{-68}$ s^2 in (7).

Recall that the denominator terms beneath $\epsilon_\circ$ and $\mu_\circ$ were somewhat arbitrarily allocated equally in the two denomitors on the right side of (5). Other arrangements are possible, but the result in (7) seems preferable because of its symmetric form. In addition, it is straightforward to modify the foregoing development to include the linear term $a_1\omega$ that was omitted from (1), in the event that future empirical data supports a non-zero linear term. In this case, (5) would become:

$$\begin{aligned} \nabla^2 \mathbf{E} &= \frac{-\omega^2}{c^2 (1 - a_1\omega - a_2\omega^2)^2} \mathbf{E} \\ &= -\frac{\mu_\circ}{(1 - a_1\omega - a_2\omega^2)} \frac{\epsilon_\circ}{(1 - a_1\omega - a_2\omega^2)} \omega^2 \mathbf{E}, \end{aligned} \tag{8}$$

with corresponding dispersive equations, along the lines of the previous development. However, recent gamma-ray results in [6] seem to favor the form in (5) rather than (8).

III. Comparison to Metamaterial Models

The dispersive gamma-ray model in (7) can be shown to be similar to certain metamaterial models. Because of this similarity, prior results for the metamaterial models offer insight to the behavior of the proposed gamma-ray dispersion. Thus, the gamma-ray model is expected to exhibit properties such as left-handed frequency bands, right-handed frequency bands, and forbidden stop-bands. In the following, prior results are first summarized, and then the the behavior of the gamma-ray model is discussed in light of the earlier metamaterial results.

Metamaterials are often modeled as composite right/left-handed (CRLH) transmission-line structures, which exhibit right-handed and left-handed behavior in frequency bands commonly separated by a forbidden frequency band, or stop band [9]. These structures are not unique, and a variety of metamaterial transmission line models can be obtained with different arrangements of series reactance and shunt reactance for the distributed transmission-line parameters.

Among such transmission line models of metamaterials, the gamma-ray model in (7) closely resembles the following CRLH model having right-handed behavior at low frequency, left-handed behavior at high frequency, and a forbidden frequency band, or stop band [7]:

$$\begin{aligned} \nabla \times \mathbf{E} + \mu\epsilon_L \frac{\partial^2}{\partial t^2} (\nabla \times \mathbf{E}) &= -\mu \frac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \mathbf{H} + \mu_L\epsilon \frac{\partial^2}{\partial t^2} (\nabla \times \mathbf{H}) &= \epsilon \frac{\partial \mathbf{E}}{\partial t}, \end{aligned} \tag{9}$$

where μ is permeability in H/m, ϵ is permittivity in F/m, ϵ_L is defined as left-permittivity in F·m, and μ_L is defined as left-permeability in H·m. For a frequency ω, equation (9) results in:

$$\begin{aligned} \nabla \times \mathbf{E} - \mu\epsilon_L \omega^2 (\nabla \times \mathbf{E}) &= -j\omega\mu \mathbf{H} \\ \nabla \times \mathbf{H} - \mu_L\epsilon \omega^2 (\nabla \times \mathbf{H}) &= j\omega\epsilon \mathbf{E}. \end{aligned} \tag{10}$$

By comparison, note that the metamaterial model in (10) is identical to the gamma-ray dispersion result in (7) when $\mu = \mu_\circ$, $\epsilon = \epsilon_\circ$, and $a_2 = \mu_L\epsilon_\circ = \mu_\circ\epsilon_L$. Thus, the behavior of the metamaterial result offers insight into the behavior of the gamma-ray dispersion result.

In addition, observe that the proposed dispersive equations in (7) and (10) revert to the normal form of Maxwell's equations in (4) at low frequencies, with a right-handed electromagnetic wave solution. In essence, the additional terms in equations (7) and (10) behave as dispersive extensions of Maxwell's equations. At extremely high frequencies, the second term on the left side of (7) dominates, and the solution becomes a dispersive left-handed electromagnetic wave [7], similar to other dispersive relationships found in metamaterials, optics, and left-handed microwave structures [9]–[11].

As noted above, the metamaterial model in (10) is right-handed at low frequency and left-handed at high frequency, with a forbidden frequency band, or stop band, between. To see this, first find the wave equation by taking the curl of both sides of line 1 of (10) (with $\nabla \cdot \mathbf{E} = 0$):

$$\nabla \times \nabla \times \mathbf{E} + \mu\epsilon_L \, \frac{\partial^2}{\partial t^2} \left(\nabla \times \nabla \times \mathbf{E} \right) = -\mu \, \frac{\partial}{\partial t} \left(\nabla \times \mathbf{H} \right),$$

or

$$\nabla^2 \mathbf{E} + \mu\epsilon_L \, \frac{\partial^2}{\partial t^2} \left(\nabla^2 \mathbf{E} \right) = \mu \, \frac{\partial}{\partial t} \left(\nabla \times \mathbf{H} \right). \tag{11}$$

Then, take $\partial/\partial t$ on both sides of line 2 of (10) and multiplying both sides by μ

$$\mu \tfrac{\partial}{\partial t} \nabla \times \mathbf{H} + \mu_L \epsilon \mu \, \tfrac{\partial^3}{\partial t^3} \left(\nabla \times \mathbf{H} \right) = \mu\epsilon \, \tfrac{\partial^2}{\partial t^2} \mathbf{E} \, . \tag{12}$$

Next, substitute the right side of (11) for the two corresponding left-hand terms of (12) to obtain the wave equation corresponding to (10) :

$$\begin{aligned} \nabla^2 \mathbf{E} \;&+\; \mu\epsilon_L \, \frac{\partial^2}{\partial t^2} \left(\nabla^2 \mathbf{E} \right) \\ &+\; \mu_L \epsilon \, \frac{\partial^2}{\partial t^2} \left(\nabla^2 \mathbf{E} + \mu\epsilon_L \, \frac{\partial^2}{\partial t^2} \left(\nabla^2 \mathbf{E} \right) \right) \\ &=\; \mu\epsilon \, \frac{\partial^2}{\partial t^2} \mathbf{E} \, , \end{aligned} \tag{13}$$

or

$$\begin{aligned} \nabla^2 \mathbf{E} + \left(\mu\epsilon_L + \mu_L \epsilon \right) \, \tfrac{\partial^2}{\partial t^2} \left(\nabla^2 \mathbf{E} \right) + \mu\epsilon\mu_L\epsilon_L \, \tfrac{\partial^4}{\partial t^4} \left(\nabla^2 \mathbf{E} \right) \\ = \mu\epsilon \, \tfrac{\partial^2}{\partial t^2} \mathbf{E} \, . \end{aligned} \tag{14}$$

Then, to find the plane-wave solution for (17), substitute $\mathbf{E} = E_\circ e^{-jkz} e^{j\omega t} \hat{x}$ to obtain:

$$\begin{aligned} (-jk)^2 \mathbf{E} \;&+\; \left(\mu\epsilon_L + \mu_L \epsilon \right) \, (j\omega)^2 (-jk)^2 \mathbf{E} \\ &+\; \mu\epsilon\mu_L\epsilon_L \, (j\omega)^4 (-jk)^2 \mathbf{E} = \mu\epsilon \, (j\omega)^2 \mathbf{E} \, , \end{aligned} \tag{15}$$

so

$$-k^2 + \left(\mu\epsilon_L + \mu_L \epsilon \right) \omega^2 k^2 - \mu\epsilon\mu_L\epsilon_L \, \omega^4 k^2 = -\mu\epsilon \, \omega^2 \, ,$$

or

$$\frac{\omega^2}{k^2} = \frac{1 - \left(\mu\epsilon_L + \mu_L \epsilon \right) \omega^2 + \mu\epsilon\mu_L\epsilon_L \, \omega^4}{\mu\epsilon} \, ,$$

then

$$\frac{\omega^2}{k^2} = \frac{\left(1 - \omega^2 \mu\epsilon_L \right) \left(1 - \omega^2 \mu_L \epsilon \right)}{\mu\epsilon} \, ,$$

and finally,

$$u^2 = \frac{\omega^2}{k^2} = \frac{\left(1 - \left(\frac{\omega}{\omega_1} \right)^2 \right) \left(1 - \left(\frac{\omega}{\omega_2} \right)^2 \right)}{\mu\epsilon} \, , \tag{16}$$

where u is phase velocity as before, $\omega_1 = 1/\sqrt{\mu\epsilon_L}$, and $\omega_2 = 1/\sqrt{\mu_L\epsilon}$.

For low frequencies where $\omega \ll \omega_1$ and $\omega \ll \omega_2$, (16) gives the normal nearly-constant right-handed velocity governed by $u^2 \approx 1/(\mu\epsilon)$. Taking the positive root, the phase velocity becomes $u = \omega/k \approx 1/\sqrt{\mu\epsilon}$, and the wavenumber becomes $k = \omega\sqrt{\mu\epsilon}$. The group velocity is then $v_g = \partial\omega/\partial k = 1/\sqrt{\mu\epsilon}$. And so, the system (9) is right-handed at low frequency, since v_g and u have the same sign [9].

At high frequencies where $\omega \gg \omega_1$ and $\omega \gg \omega_2$, (16) gives a highly dispersive left-handed velocity set by $u^2 \approx \omega^4 \mu_L \epsilon_L$. Taking the positive root, the phase velocity becomes $u = \omega/k \approx \omega^2 \sqrt{\mu_L\epsilon_L}$, and the wavenumber becomes $k = 1/(\omega\sqrt{\mu_L\epsilon_L})$. The group velocity is then $v_g = \partial\omega/\partial k = -1/(k^2\sqrt{\mu_L\epsilon_L})$. And so, the system (9) is left-handed at high frequency, since v_g and u have the opposite sign [9]. In addition, there is a forbidden band between ω_1 and ω_2 where u^2 is negative, k becomes imaginary, and the solution is evanescent and does not propagate [7].

Again comparing the metamaterial model of (10) with the gamma-ray model of (7), the two results are identical when $\mu = \mu_\circ$, $\epsilon = \epsilon_\circ$, and $a_2 = \mu_\circ\epsilon_L = \mu_L\epsilon_\circ \approx 1.74 \times 10^{-68}$ s^2. Then the velocity in (16) becomes

$$u^2 \;\; = \;\; \frac{\left(1 - a_2 \omega^2 \right)^2}{\mu_\circ \epsilon_\circ} = c^2 \left(1 - a_2 \omega^2 \right)^2 , \tag{17}$$

which equals the square of the gamma-ray result in (2). Note also that $a_2 = \mu_\circ\epsilon_L = \mu_L\epsilon_\circ$ implies that $\omega_1 = \omega_2 = 1/\sqrt{\mu_L\epsilon} = 1/\sqrt{\mu_\circ\epsilon_L}$. In this case, the forbidden band consists of the single frequency, $\omega_1 = \omega_2$.

Finally, the gamma-ray model in (7) has the same form as the metamaterial model in (10), and therefore will exhibit the same general behavior. Thus, the gamma-ray model is expected to show right-handed behavior at low frequency and

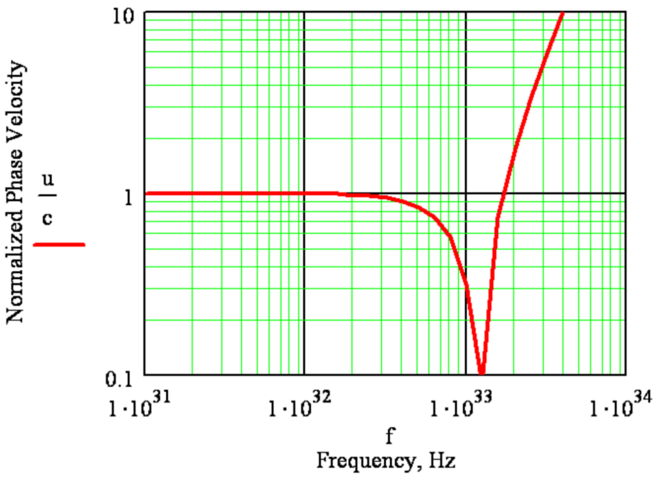

Fig. 1. Plot of phase velocity u normalized to the speed of light c as a function of frequency in Hz.

left-handed behavior at high frequency. For the gamma-ray model, the stop-band is expected to be a single frequency separating the right-handed frequency band from the left-handed band. In particular, the stop-band frequency based on measured gamma-ray data is $\omega_1 = \omega_2 = 1/\sqrt{\mu_L \epsilon} = 1/\sqrt{\mu_\circ \epsilon_L} = 1/\sqrt{a_2} \approx 1/\sqrt{1.74 \times 10^{-68}} = 7.58 \times 10^{33}$ rad/s. For the gamma-ray model (7), taking the positive root of (16) the phase velocity becomes:

$$u = \frac{\omega}{k} = \left| c \left(1 - \left(\frac{\omega}{7.58 \times 10^{33}} \right)^2 \right) \right| , \tag{18}$$

and is plotted in Fig. 1.

As illustrated in Fig. 1, the phase velocity u is a nearly constant value of c for all frequencies of practical interest. And so, it is only over astronomical distances that measurable time delays are observed, even for a 31 GeV photon with a frequency of 7.49×10^{24} Hz in gamma-burst GRB 090510 [6]. At high frequencies approaching the Planck frequency of 2.95×10^{42} Hz (corresponding to the Planck energy of $\mathcal{E}_p = 1.22 \times 10^{28}$ eV), controversy over fundamental physics remains [5]. And since the starting point in (1) is motivated by a power-series approximation, care must be taken in any interpretation of behavior at frequencies beyond the estimated resonance at $\omega_1 = \omega_2 \approx 7.58 \times 10^{33}$.

IV. Conclusion

The dispersion observed in gamma-ray bursts suggests the need for corresponding adjustments to the Helmholtz equation. Even though the underlying dispersive mechanisms are not well understood, previous investigators have focused on a quadratic dispersion model, since experimental data disfavor a linear model. Thus, a modified Helmholtz equation with quadratic dispersion is presented to model propagation of gamma rays. Although several forms of the equations are possible, one form was chosen because of its symmetry. In addition, the resulting added terms in the equations act as dispersive extensions to Maxwell's equations, preserving normal behavior at low frequency while supporting dispersion for high-frequency gamma rays. Ultimately, the choice of the proper form of the Maxwell equations and the Helmholtz equation may change as more experimental results on gamma-ray bursts become available.

References

[1] L. Baldini, "The commissioning and first light of the fermi large area telescope," *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, vol. 604, no. 1-2, pp. 164 – 167, 2009, pSD8 - Proceedings of the 8th International Conference on Position Sensitive Detectors. [Online]. Available: http://www.sciencedirect.com/science/article/B6TJM-4VGWM9V-B/2/864473ad3d0f7104540f06c83ba0ed7a

[2] L. Shao, Z. Xiao, and B.-Q. Ma, "Lorentz violation from cosmological objects with very high energy photon emissions," *Astroparticle Physics*, vol. 33, no. 5-6, pp. 312 – 315, 2010. [Online]. Available: http://www.sciencedirect.com/science/article/B6TJ1-4YM7FFJ-2/2/f45b636911604cc4a2f51a086b0a858c

[3] J. Albert, E. Aliu, H. Anderhub, L. Antonelli, and et al., "Probing quantum gravity using photons from a flare of the active galactic nucleus markarian 501 observed by the magic telescope," *Physics Letters B*, vol. 668, no. 4, pp. 253 – 257, 2008. [Online]. Available: http://www.sciencedirect.com/science/article/B6TVN-4TBXGBD-2/2/5b509a6a83bf5cca1d6e272d1efc7e69

[4] F. Aharonian, A. G. Akhperjanian, U. Barres de Almeida, and et al., "Limits on an energy dependence of the speed of light from a flare of the active galaxy pks 2155-304," *Phys. Rev. Lett.*, vol. 101, no. 17, p. 170402, Oct 2008.

[5] G. Amelino-Camelia, J. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, "Tests of quantum gravity from observations of [gamma]-ray bursts," *Nature*, vol. 393, pp. 763–765, 1998. [Online]. Available: http://www.nature.com/nature/journal/v393/n6687/abs/393763a0.html

[6] A. Abdo, M. Ackermann, M. Ajello, and et al., "A limit on the variation of the speed of light arising from quantum gravity effects," *Nature*, vol. 462, pp. 331–334, 2009. [Online]. Available: http://www.nature.com/nature/journal/v462/n7271/abs/nature08574.html

[7] T. Weldon, R. Adams, K. Daneshvar, and R. Mulagada, "Left-handed extensions of maxwell's equations for metamaterials," in *IEEE SoutheastCon 2010 (SoutheastCon), Proc. of the*, Charlotte, NC, USA, Mar. 2010, pp. 489–492.

[8] T. V. D. S. Ramo, J.R. Whinnery, *Fields and Waves in Communication Eelctronics, 2nd Ed.,*. New York: John Wiley and Sons, 1984.

[9] A. Lai, T. Itoh, and C. Caloz, "Composite right/left-handed transmission line metamaterials," *IEEE Microw. Mag.*, vol. 5, no. 3, pp. 34– 50, Sep. 2004.

[10] N. Engheta and R. Ziolkowski, "A positive future for double-negative metamaterials," *IEEE Trans. Microw. Theory Tech.*, vol. 53, no. 4, pp. 1535–1556, Apr. 2005.

[11] K. Sinchuk, , R. Dudley, J. D. Graham, M. Clare, M. Woldeyohannes, J. O. Schenk, R. P. Ingel, W. Yang, and M. A. Fiddy, "Tunable negative group index in metamaterial structures with large form birefringence," *Optics Express*, vol. 18, no. 2, pp. 463–472, 2010. [Online]. Available: http://www.opticsinfobase.org/abstract.cfm?URI=oe-18-2-463