Left-Handed Extensions of Maxwell's Equations for Metamaterials

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Abstract— **Left-handed materials are commonly analyzed by using frequency-dependent parameters that include regions of negative permittivity and negative permeability in Maxwell's equations. An alternative approach is presented where both right-handed and left-handed behavior are supported by augmenting Maxwell's equations with additional terms. The proposed equations exhibit typical metamaterial behavior, including frequency bandgap and dispersion.**

I. Introduction

Recent advances in left-handed materials and negative index materials have motivated considerable research in the development of new optical devices and novel microwave circuits. For the most part, left-handed phenomena have been studied using familiar forms of Maxwell's equations, but with negative values of permittivity and permeability at frequencies of interest [1]-[2]. With this approach, the analysis of left-handed systems may be carried out using Maxwell's equations in conjunction with frequency-dependent (or resonant) parameters for permittivity and permeability, where the parameters become negative at certain frequencies.

However, it is also possible to produce a frequency-dependent model by modifying the differential equations and using a new set of constant parameters. In the proposed approach, the frequency dependence is built into the form of the differential equations rather than prior methods relying upon frequency-dependent parameters for permittivity and permeability. Although the new approach may be equivalent to prior methods in various situations, the proposed line of attack seems to offer new insight into the behavior and theory of left-handed systems.

In the proposed model, the new left-handed extensions of Maxwell's equations are developed using an approach that is motivated by recent developments for left-handed microwave systems [3]-[4]. Here, transmission line circuits are used as a guide to formulate corresponding modifications to Maxwell's equations. Although multiple transmission line circuit topologies are possible, the development focuses on a form that is right-handed at low frequencies and left-handed at high frequencies. Using the new equations, some general observations are made on predicted behavior, including frequency bandgap and dispersion.

II. Right-Handed Maxwell's Equations

A useful framework for the development of the proposed extensions to Maxwell's equations is found in recent research on composite right/left-handed (CRLH) microwave structures [3]. In this earlier work, lumped-element transmission lines are used as the basis for the formulation of new microwave structures and theoretical analysis. Importantly, these left-handed microwave transmission line models use constant parameters that are not frequency dependent.

The success of such theoretical methods in generating fruitful results for left-handed microwave systems motivates the proposed left-handed extensions to Maxwell's equations for metamaterials and free space. In the following, the correspondence between transmission line equations and Maxwell's equations is first reviewed for right-handed systems. Then, these results are used as a template for developing the proposed left-handed extensions of Maxwell's equations in the following section.

To begin, first consider the right-handed transmission line model of Fig. 1. The right-handed transmission line equations for Fig. 1 also correspond to the three-dimensional Maxwell's equations as follows [6] (assuming no sources):

$$\begin{aligned} \frac{\partial v(x,t)}{\partial x} &= -L_R \frac{\partial i(x,t)}{\partial t} \quad \Rightarrow \quad \nabla \times E = -\mu \frac{\partial H}{\partial t} \\ \frac{\partial i(x,t)}{\partial x} &= -C_R \frac{\partial v(x,t)}{\partial t} \quad \Rightarrow \quad \nabla \times H = \varepsilon \frac{\partial E}{\partial t} \end{aligned} \quad , (1)$$

where the transmission line equations are on the left and the corresponding Maxwell's equations are on the right. The transmission line distributed inductance is L_R H/m and distributed capacitance is C_R F/m, and the Maxwell's equation constants are μ for the permeability in H/m and ε for the permittivity in F/m.

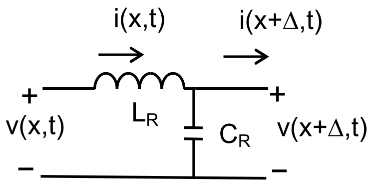

Figure 1. Lumped-element model of a right-handed transmission line.

Taking the curl of both sides from (1):

$$\nabla \times \nabla \times E = -\mu \frac{\partial \nabla \times H}{\partial t} \quad . \tag{2}$$

Then, the wave equation follows as (where $\nabla \cdot E=0$):

$$\nabla^2 E = \mu\varepsilon \frac{\partial^2 E}{\partial t^2} \quad , \tag{3}$$

with the usual plane-wave solution

$$E = E_0 e^{-jkx} e^{j\omega t} = E_0 e^{-j\omega x/u} e^{j\omega t} \quad , \tag{4}$$

where k=ω/u is the wavenumber, ω is frequency in rad/s, and $u=(\mu\varepsilon)^{-1/2}$ is the phase velocity in m/s.

III. PROPOSED EXTENSION OF MAXWELL'S EQUATIONS

The foregoing correspondence between Fig. 1 and Maxwell's equations in (1) for right-handed systems is next used as a template for developing left-handed extensions to Maxwell's equations for metamaterials. Along the lines of Fig. 1, a composite right/left-handed transmission line is given in Fig. 2. The right-handed distributed parameters are L_R in H/m and C_R in F/m, and the left-handed distributed parameters are L_L in H·m and C_L in F·m.

The topology in Fig. 2 exhibits right-handed behavior at low frequencies and exhibits left-handed behavior above the frequency bandgap. The corresponding extension of Maxwell's equations will then exhibit the normal right-handed behavior of Maxwell's equations at low frequency. Although not considered here, the general methodology can also be used for other transmission line topologies along the lines of Fig. 2, where such topologies may be better suited to model other particular problems or metamaterials.

Following the same approach used in equation (1) and Fig. 1, the composite right/left-handed transmission line equations for Fig. 2 correspond to the new proposed extension of three-dimensional Maxwell's equations as follows [3],[6] (assuming no sources):

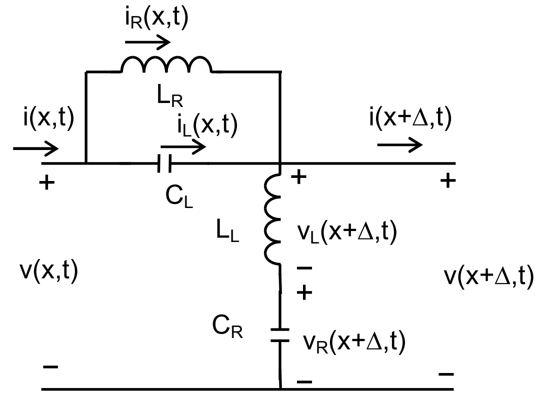

Figure 2. Lumped-element model of a right/left-handed transmission line.

$$\begin{aligned}
&\frac{\partial v(x,t)}{\partial x} = -L_R \frac{\partial i_R(x,t)}{\partial t} &&\Rightarrow \quad \nabla \times E = -\mu \frac{\partial}{\partial t} H_R \\
&\frac{\partial v(x,t)}{\partial x} = -\frac{1}{C_L} \int i_L(x,t)\partial t &&\Rightarrow \nabla \times E = -\frac{1}{\varepsilon_L} \int H_L \partial t \\
& && or \quad \frac{\partial}{\partial t} \nabla \times E = -\frac{1}{\varepsilon_L} H_L \\
&\frac{\partial i(x,t)}{\partial x} = -C_R \frac{\partial v_R(x,t)}{\partial t} &&\Rightarrow \quad \nabla \times H = \varepsilon \frac{\partial}{\partial t} E_R \\
&\frac{\partial i(x,t)}{\partial x} = -\frac{1}{L_L} \int v_L(x,t)\partial t &&\Rightarrow \quad \nabla \times H = \frac{1}{\mu_L} \int E_L \partial t \\
& && or \quad \frac{\partial}{\partial t} \nabla \times H = \frac{1}{\mu_L} E_L \\
&where \quad v = v_R + v_L &&\Rightarrow \quad E = E_R + E_L \\
&and \quad i = i_R + i_L &&\Rightarrow \quad H = H_R + H_L
\end{aligned} \quad ,(5)$$

where the transmission line equations are on the left and the corresponding proposed extensions to Maxwell's equations are on the right. Derivatives are also taken in two places to remove the integral forms in the 3rd and 6th equations of (5). On the right side of (5), μ is permeability in H/m, ε is permittivity in F/m, ε_L is defined as left-permittivity in F·m, and μ_L is defined as left-permeability in H·m. In addition, the electric (*E*) and magnetic (*H*) fields are expressed in terms of the sums of left-handed contributions (E_L and H_L) and right-handed contributions (E_R and H_R) to the total fields, in the same manner as the decomposition of the currents and voltages of Fig. 2 into left-handed (v_L and i_L) and right-handed (v_R and i_R) components.

The equations in (5) can be combined to eliminate the left-handed and right-handed components of the fields. Taking the derivative of one of the equations from (5) and adding the result to another one of the equations in (5), H_R and H_L can be eliminated as follows:

$$\frac{\partial}{\partial t}\left(\varepsilon_L \frac{\partial}{\partial t}\nabla\times E\right) = \frac{\partial}{\partial t}(-H_L)$$

$$\text{so} \qquad \varepsilon_L \frac{\partial^2}{\partial t^2}\nabla\times E = -\frac{\partial}{\partial t}H_L \qquad (6)$$

$$\text{and} \qquad \frac{1}{\mu}\nabla\times E = -\frac{\partial}{\partial t}H_R \qquad \text{from (5)}$$

$$\text{adding} \;\Rightarrow\; \frac{1}{\mu}\nabla\times E + \varepsilon_L \frac{\partial^2}{\partial t^2}\nabla\times E = -\frac{\partial}{\partial t}(H_R + H_L) = -\frac{\partial}{\partial t}H \,.$$

Similarly, E_R and E_L can be eliminated:

$$\frac{\partial}{\partial t}\left(\mu_L \frac{\partial}{\partial t}\nabla\times H\right) = \frac{\partial}{\partial t}(E_L)$$

$$\text{so} \qquad \mu_L \frac{\partial^2}{\partial t^2}\nabla\times H = \frac{\partial}{\partial t}E_L \qquad (7)$$

$$\text{and} \qquad \frac{1}{\varepsilon}\nabla\times H = \frac{\partial}{\partial t}E_R \qquad \text{from (5)}$$

$$\text{adding} \;\Rightarrow\; \frac{1}{\varepsilon}\nabla\times H + \mu_L \frac{\partial^2}{\partial t^2}\nabla\times H = \frac{\partial}{\partial t}(E_R + E_L) = \frac{\partial}{\partial t}E \,.$$

The foregoing results give the final proposed extension of Maxwell's equations for the right/left-handed system (assuming no sources):

$$\begin{aligned} &\frac{1}{\mu}\nabla\times E + \varepsilon_L \frac{\partial^2}{\partial t^2}\nabla\times E = -\frac{\partial}{\partial t}H \\ &\frac{1}{\varepsilon}\nabla\times H + \mu_L \frac{\partial^2}{\partial t^2}\nabla\times H = \frac{\partial}{\partial t}E \end{aligned} \;. \qquad (8)$$

Unfortunately, the associated wave equation for the right/left-handed system of (8) is not straightforward, as seen by taking the curl of both sides of (8):

$$\frac{1}{\mu}\nabla\times\nabla\times E + \varepsilon_L \frac{\partial^2}{\partial t^2}\nabla\times\nabla\times E = -\frac{\partial}{\partial t}\nabla\times H \,. \quad (9)$$

Then, the wave equation follows as (where $\nabla\cdot E=0$):

$$\frac{1}{\mu}\nabla^2 E + \varepsilon_L \frac{\partial^2}{\partial t^2}\nabla^2 E = \varepsilon \frac{\partial^2}{\partial t^2}E_R \;, \qquad (10)$$

where the right side of the result is complicated by the appearance of E_R instead of E. Nevertheless, it is possible to solve for E. Suppose a plane wave solution for E as follows:

$$E = E_0 e^{-jkx} e^{j\omega t} = E_0 e^{-j\omega x/u} e^{j\omega t} \;. \qquad (11)$$

Substituting the plane wave solution, and after considerable rearrangement, the solution for the square of the phase velocity u^2 is:

$$u^2 = \frac{(1-\omega^2\mu\varepsilon_L)(1-\omega^2\varepsilon\mu_L)}{\mu\varepsilon} = \frac{1}{\mu\varepsilon}\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right), \quad (12)$$

where the frequency bandgap is determined by $\omega_1=(\mu_L\varepsilon)^{-1/2}$ and $\omega_2=(\mu\varepsilon_L)^{-1/2}$.

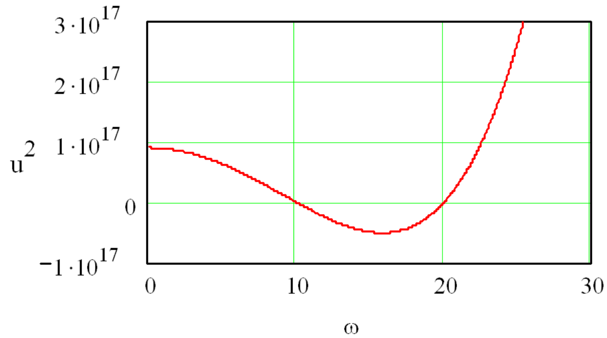

Figure 3. Plot of square of phase velocity, u^2, as a function of frequency, ω, for $(\mu\varepsilon)^{-1/2}=3\times10^8$, $\omega_1=10$ and $\omega_2=20$ rad/s.

From (12), the phase velocity is $u\approx(\mu\varepsilon)^{-1/2}$ at low frequency. Between ω_1 and ω_2 there is a bandgap where u^2 is negative, the phase velocity is imaginary, and the wave exponentially decays without propagating. And at higher frequencies beyond the bandgap, a left-handed solution is found with dispersive phase velocity $u\approx\omega^2(\mu_L\varepsilon_L)^{1/2}$. To illustrate the behavior of (12), u^2 is plotted in Fig. 3 for a somewhat arbitrary case of $(\mu\varepsilon)^{-1/2}=c=3\times10^8$, $\omega_1=10$, and $\omega_2=20$ rad/s. In Fig. 3, negative values of u^2 fall between the bandgap frequencies of $\omega_1=10$ and $\omega_2=20$ rad/s.

In addition, the ratio of the left-handed field to right-handed field can be calculated. To find this ratio, first use the following relation from (5):

$$\varepsilon\frac{\partial}{\partial t}E_R = \nabla\times H = \frac{1}{\mu_L}\int E_L \partial t \;, \qquad (13)$$

so

$$\varepsilon\frac{\partial^2}{\partial t^2}E_R = \frac{1}{\mu_L}E_L \qquad (14)$$

and then $E_L=-\varepsilon\mu_L\omega^2 E_R$ for the plane wave solution as before. Therefore, the ratio of the left-handed field contribution E_L to the right-handed field contribution E_R would be approximately $|E_L/E_R| = \varepsilon\mu_L\omega^2$.

Interestingly, the bandgap frequency estimate from recent gamma ray data can be used to estimate the degree to which $\mu_L\to0$ and $\varepsilon_L\to0$ in (8) for free space [5]. Note that when $\mu_L=0$ and $\varepsilon_L=0$, the extended Maxwell equations in (8) revert to the normal Maxell's equations in (1). Recent data on gamma burst GRB 090510 in May 2009 shows dispersion of up to 859 ms for gamma rays at 31 GeV ($\omega=4.7\times10^{25}$ rad/s) at a distance of $d=1.8\times10^{26}$ m (using luminosity distance for simplicity). If 31 GeV photons have velocity u, and low energy photons have velocity c, the time difference in arrival times over a distance d is $\tau=d/u-d/c$ (ignoring cosmological issues for simplicity). After some rearrangement, the velocity of the 31 GeV photons is approximated as

$$u = c\frac{d}{d+c\tau} = c\frac{d^2-cd\tau}{d^2-c^2\tau^2} \approx c\left(1-\frac{c\tau}{d}\right), \quad (15)$$

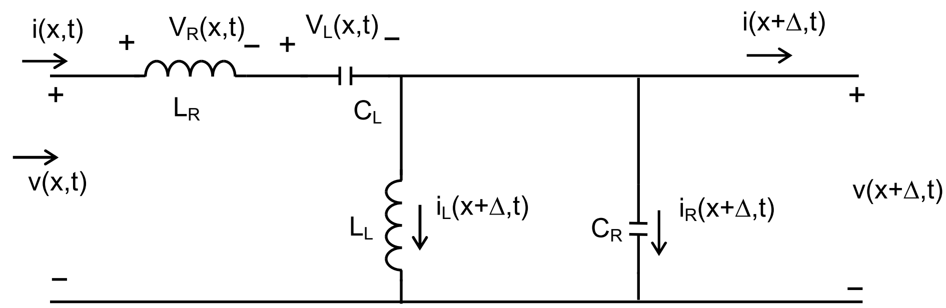

Figure 4. Lumped-element model of an alternative form of a right-/left handed transmission line.

where $c\tau << d$, $c=3\times10^8$ m/s, $\tau=0.859$ s, $d=1.8\times10^{26}$ m, and $c\tau/d=1.4\times10^{-18}$. Since (12) has two remaining unknowns, further approximations are used to provide an estimate of the lower bandgap frequency. For simplicity, let $\omega_1=\omega_2$, where equation (12) becomes $u=c\{1-(\omega/\omega_1)^2\}$ in free space. Comparing this result with (15) gives $(\omega/\omega_1)^2=c\tau/d=1.4\times10^{-18}$ for the 31 GeV photon at a frequency of $\omega=4.7\times10^{25}$ rad/s. Then solving for ω_1, the estimate of the bandgap frequency is $\omega_1\approx3.9\times10^{34}$ rad/s or 2.6×10^{19} eV. Finally, using the estimate of $\omega_1=\omega_2=3.9\times10^{34}$ rad/s and substituting into the equations $\omega_1=(\mu_L\varepsilon)^{-1/2}$ and $\omega_2=(\mu\varepsilon_L)^{-1/2}$ from (12) results in $\mu_L\approx1/(\varepsilon_0\omega_1^2)\approx7.4\times10^{-59}$ H·m and $\varepsilon_L\approx1/(\mu_0\omega_1^2)\approx 5.1\times10^{-64}$ F·m in free space, where $\varepsilon_0=8.9\times10^{-12}$ F/m and $\mu_0=1.3\times10^{-6}$ H/m. Thus, $\mu_L\rightarrow0$ and $\varepsilon_L\rightarrow0$ in (8) for free space, for most frequencies of practical interest.

IV. ALTERNATE FORMS

As previously mentioned, the forgoing methodology can be applied to circuit topologies other than Fig. 2. A more common form is the composite right/left hand (CRLH) structures of Fig. 4, described in [3]. The band-pass topology of Fig. 4 is similar to the band-stop topology of Fig. 2, and is left-handed at low frequencies and right-handed at high frequencies. For the topology of Fig. 4, the left-handed extensions of Maxwell's equations yield:

$$\begin{aligned}\frac{\partial}{\partial t}\nabla\times E &= -\mu\frac{\partial^2}{\partial t^2}H-\frac{1}{\varepsilon_L}H\\ \frac{\partial}{\partial t}\nabla\times H &= \varepsilon\frac{\partial^2}{\partial t^2}E+\frac{1}{\mu_L}E\end{aligned}\quad, \qquad (16)$$

where μ, ε, ε_L, and μ_L are defined as before. Then, the solution for the square of the plane-wave phase velocity u^2 is:

$$u^2=\frac{\omega^4\mu_L\varepsilon_L}{\left(1-\omega^2\mu\varepsilon_L\right)\left(1-\varepsilon\mu_L\omega^2\right)}=\frac{\omega^4\mu_L\varepsilon_L}{\left(1-(\omega/\omega_1)^2\right)\left(1-(\omega/\omega_2)^2\right)}\ ,(17)$$

where the frequency bandgap is determined by $\omega_1=(\mu_L\varepsilon)^{-1/2}$ and $\omega_2=(\mu\varepsilon_L)^{-1/2}$. From (16), the phase velocity is $u\approx(\mu\varepsilon)^{-1/2}$ at high frequency, above the bandgap. Between ω_1 and ω_2 there is a bandgap where u^2 is negative. And at frequencies below the bandgap, a left-handed solution is found with dispersive phase velocity $u\approx\omega^2(\mu_L\varepsilon_L)^{1/2}$.

V. CONCLUSION

The proposed extensions to Maxwell's equations incorporate familiar frequency-independent permittivity and permeability constants and add new frequency-independent left-handed permittivity and left-handed permeability constants. These four constants, along with the new equations, lead to frequency-dependent solutions with a bandgap, frequency-dependent phase velocity, and dispersion. This may be somewhat unexpected, since constant permeability and permittivity give rise to a constant velocity in the conventional form of Maxwell's equations. In essence, the new proposed extension to Maxwell's equations incorporates the left-handed and right-handed behavior into the form of the new equations, rather than using frequency-dependent permittivity and permeability to incorporate left-handed behavior. Although two particular forms of left-handed extensions of Maxwell's equations are given, the methodology can be applied to a variety of other circuit topologies that may be suitable for different metamaterial applications.

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