Measured and Field-Based Theoretical Q of Gravitationally-Small Gravitational Antennas

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Abstract—**Recent breakthroughs in detection of gravitational waves not only demonstrate the transmission and reception of such waves, but also prompt comparisons to electromagnetic antennas. Although the transmitting gravitational antenna and signal generator may be the inspiral of a binary neutron star orbit, such new gravitational antenna observations raise interesting questions from an antenna engineering perspective. In particular, measured data are used to determine the Q (quality factor) of the gravitationally-small gravitational antenna comprised of the neutron star quadrupole, along the lines of Wheeler-Chu limits for electromagnetic antennas. The recent gravitational wave event GW170817 had signal frequencies below 75 Hz and implied orbital separation near 250 km, which suggest a gravitationally-small radiator below 0.07 wavelengths. In addition, qualitative arguments based on quadrupole gravitational field components suggest a theoretical Q inversely proportional to the seventh power of gravitational quadrupole antenna size. Measured data are presented that somewhat better support quadrupole gravitational antenna Q inversely proportional to the fifth power of gravitational antenna size, more so than seventh power. Lastly, an antenna engineering perspective should provide insight for metamaterial and non-Foster enhancement of gravitational antenna designs.**

I. INTRODUCTION

The last few years have seen tremendous progress in the physics of gravity, particularly with the LIGO (Laser Interferometer Gravitational-Wave Observatory) detection of gravitational waves from binary black hole merger in GW150914 and binary neutron star merger in GW170817 [1], [2]. The overall scenario of gravitational wave transmission requires ”gravitational antennas” at transmitter and receiver, even if the transmit antenna is a binary neutron star powered by their inspiral and coalescence. From an antenna engineering perspective, it is apparent that the GW170817 event with frequencies below 75 Hz is a gravitationally-small gravitational antenna, with an approximately 250 km orbit dimension that would correspond to less than 0.06 wavelength, similar to a $ka = 0.39$ dimension parameter for the Wheeler-Chu limit of an electrically-small antenna [3].

Therefore, GW170817 frequency chirp data is used to measure Q variation for the binary neutron star gravitational antenna, with particular interest in best fits to integer power laws such as $1/(ka)^5$ or $1/(ka)^7$. In addition to measured Q, gravitoelectric field theory in [4] is used to qualitatively predict that the Q of gravitational quadrupoles should be of the order of $O(1/(ka)^7)$. Despite the use of coarse approximations to general relativity in the analysis, the measured results include methods and experimental plots providing *measured Q of gravitational antennas* over a wide range of ka values.

Also, it should be emphasized that the theoretical Q under consideration is due to radiated and non-radiated gravitational field energy, and is distinctly different form the mechanical Q of cryogenic spherical gravitational antennas in [5], where the mechanical Q arises from mechanical damping and losses. However, these mechanical Q issues may be be viewed from an antenna engineering perspective as analogous to ohmic losses in antennas that decrease antenna efficiency. In fact, the predicted gravitational Q would be 3×10^{14} for $1/(ka)^7$ and a 4 km long terrestrial observatory at 100 Hz, versus cryogenic gravitational antenna mechanical Q of the order 10^8 due to material stiffness as noted in [5]. It would therefore seem that the gravitational Q factor based on gravitational field energy described below could present more severe design constraints.

II. ESTIMATION OF GRAVITATIONAL ANTENNA Q

The GW170817 gravitational wave observed by LIGO in August 2017 was caused by the inspiral and merger of a binary neutron star pair [2]. Without the gravitational wave energy loss, the stars would remain in their original orbits by conservation of energy. However, the orbital inspiral is a consequence of huge losses in orbital energy $E_\circ$ due to gravitational wave luminosity $\mathcal{L}$, with

$$\mathcal{L} = 32(m_1+m_2)^5\nu^2G^4/(5a_\circ^5c^5)\ , \tag{1}$$

where the binary star masses are m_1 and m_2 in kg, the separation between stars is $a_\circ$ m, $\nu = m_1m_2/(m_1+m_2)^2$, and $G = 6.7\times10^{-11}$ N·(m/kg)^2 is the gravitational constant [6].

Previously published parameters for GW170817 in [2] include star masses $m_1 \approx 3.6\times10^{30}$ kg and $m_2 \approx 2.2\times10^{30}$ kg, and include time-frequency maps of the observed frequency of the gravitational wave $\omega_{gw} = 2\omega_\circ$, where the orbital frequency is $\omega_\circ$ rad/s. The computed orbital separation $a_\circ$ follows from Kepler's Third Law, where $\omega_\circ^2 = (m_1+m_2)G/a_\circ^3$, so $a_\circ = [(m_1+m_2)G/\omega_\circ^2]^{1/3}$. Thus, Fig. 1(a) shows measured gravitational wave frequency f_{gw} in Hz estimated from 23 points of the time-frequency map in [2], and the computed orbital separation between binary stars $a_\circ$. Note that the significant change in orbital separation $a_\circ$ provides a wide range of gravitational-antenna size parameter $ka_\circ = a_\circ\omega_{gw}/c$ for gravitational wave velocity $c = 3\times10^8$ m/s, as shown in Fig. 1(b). The Newtonian orbital energy $E_\circ \approx -0.5(m_1+m_2)^2\nu G/a_\circ$

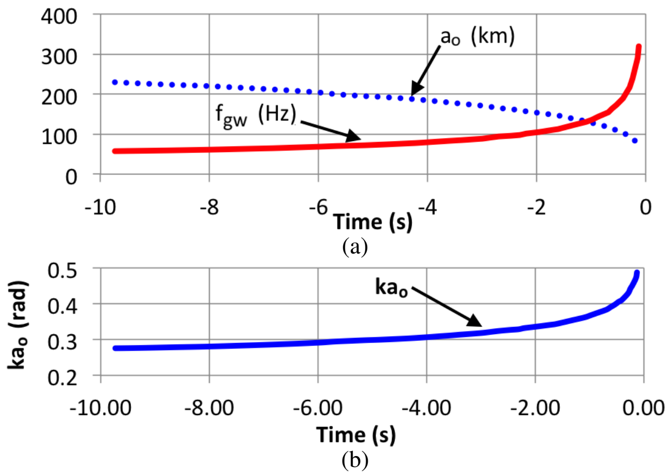

Fig. 1. Data from binary neutron star inspiral GW170817 plotted as a function of time ($t = 0$ at coalescence). (a) Measured gravitational wave frequency $f_{gw} = \omega_{gw}/(2\pi)$ in solid red is estimated from time-frequency map in [2], and the computed orbital separation $a_\circ$ is the dotted blue curve. (b) The gravitational antenna size parameter $ka_\circ = a_\circ \omega_{gw}/c$ plotted in solid blue shows a gravitationally-small antenna, where $\omega_{gw} = 2\omega_\circ$, and $\omega_\circ$ is the orbital frequency.

from [6] can be calculated using the plotted $a_\circ(t)$ in Fig. 1(a), and similarly the gravitational wave luminosity $\mathcal{L}$ also can be calculated from (1) and $a_\circ(t)$.

As shown in Fig. 1(a), the gravitational wave frequency ω_{gw} and orbital separation $a_\circ$ are slowly varying over much of the chirp duration, and so may be modeled as a damped second-order system, much like the series-RLC model of an inductively-tuned electrically-short dipole. For this electrical analog, the natural response with some initial current $i_\circ$ is governed by $Ld^2i(t)/dt^2 + Rdi(t)/dt + i(t)/C = 0$ with solution $i(t) \approx i_\circ e^{-Rt/(2L)} \cos(t/\sqrt{LC})$ for $Q \gg 1$. Expressing this in standard damped second-order form with $Q = \omega_\circ L/R$ and resonant frequency $\omega_\circ \approx 1/\sqrt{LC}$, then $i(t) \approx i_\circ e^{-0.5\omega_\circ t/Q} \cos(\omega_\circ t)$. Noting that initial energy in the system is $E_\circ = i_\circ^2 L/2$, and thus for large Q the energy decay falls in proportion to the square of the peak current, so $E_\circ(t) \approx E_\circ e^{-\omega_\circ t/Q}$ for $Q \gg 1$. Taking the derivative, $dE_\circ(t)/dt \approx (\omega_\circ E_\circ/Q)e^{-\omega_\circ t/Q}$, and solve for measured Q:

$$Q_{meas} \approx \frac{\omega_\circ E_\circ(t)}{dE_\circ(t)/dt} \, , \tag{2}$$

where Newtonian orbital energy is $E_\circ(t) \approx -0.5\nu G(m_1 + m_2)^2/a_\circ(t)$ from [6], time dependent orbital separation $a_\circ(t)$ is from Fig. 1(a), and $dE_\circ(t)/dt$ can be computed from $E_\circ(t)$ directly. The measured Q of the binary neutron star GW170817 using (2) is shown as the solid blue curve of Fig. 2, for the time-dependent range of $ka_\circ$ of Fig 1(b). The dashed red lcurve of Fig. 2 is the best integer power-law fit of $21.7/(ka_\circ)^5$, and the dotted magenta curve is the best seventh-power fit of $2.6/(ka_\circ)^7$.

Lastly, the Q-dependence of gravitationally-small antennas is inferred from the ratio of stored non-propagating near-field (r small) energy divided by propagating (r large) energy terms in the gravitational field. Analogously, a radiating electric field is comprised of E and H field components inversely proportional to the radius, or of the order $O(1/r)$. In the case of a Hertzian electric dipole, there is also a strong non-radiating near-field term of the order $O(1/r^3)$, with energy inside the radiansphere roughly of order $\approx \int |O(1/r^3)|^2 \, r^2 \sin(\theta) d\theta d\phi dr$. This integral results in near-field non-radiating energy of the order $O(1/r^3)$, leading to the cubic dependence in the Wheeler-Chu limit, as noted in [3]. For both the electromagnetic and gravitational cases, conservation of energy constrains the radiation to be associated with the order $O(1/r)$ field terms. The gravitational field of the gravitational quadrupole has non-radiating terms of the order $O(1/r^5)$ as noted in [4], leading to energy inside the radiansphere roughly of order $\approx \int |O(1/r^5)|^2 \, r^2 \sin(\theta) d\theta d\phi dr \approx O(1/r^7)$. Thus, the predicted Q-dependence for a gravitational antenna is $Q \approx 1/(ka_\circ)^7$, though the results of Fig. 2 are a better fit (for integer power laws) to $Q \approx 21.7/(ka_\circ)^5$. With the variation of Q_{meas} below $ka_\circ = 0.35$ in Fig. 2, it is difficult to tell if the slope may be steepening toward seventh order for smaller values of $ka_\circ$. More accurate data and general relativity calculations in the future may refine the foregoing coarse approximations.

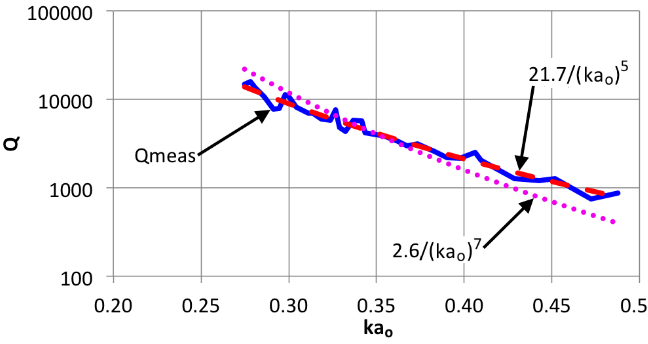

Fig. 2. Measured Q for binary neutron star inspiral GW170817 as a function of gravitational antenna size parameter $ka_\circ$, based on f_{gw} estimated from time-frequency map in [2] . The measured gravitational wave antenna quality factor Q_{meas} from (2) is in solid blue. The dashed red curve equals $21.7/(ka_\circ)^5$ best fit to an integer power law, dotted magenta curve equals $2.6/(ka_\circ)^7$ best fit to seventh-power law.

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