BEWARE OF TYPOS!

FILTER DESIGN METHODS (CHAPTER 7) OPPENHEIM.

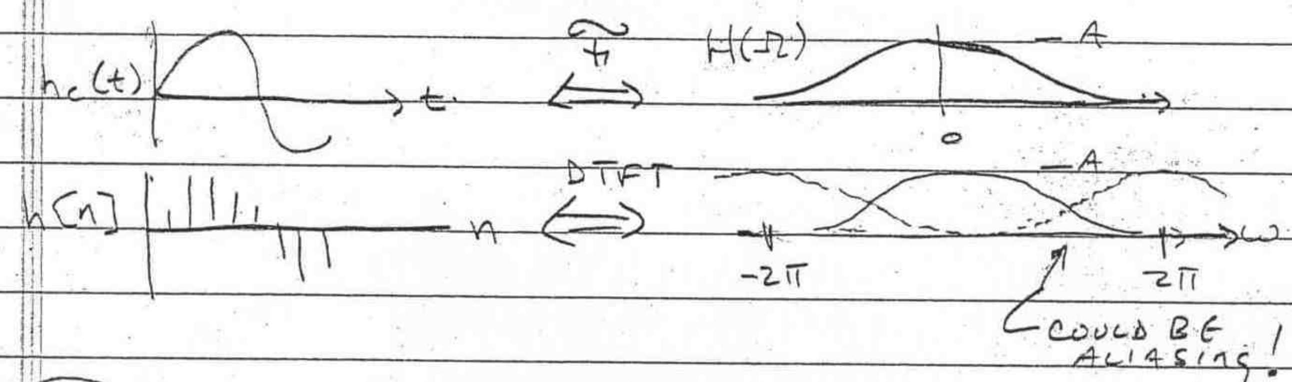
GENERALLY START WITH SOME ANALOG FILTER DESIGN, $h_c(t)$ or $H_c(s)$, AND SOMEHOW CONVERT THIS to A DIGITAL FILTER, $h[n]$ OR $H(z)$

(1) IMPULSE INVARIANCE METHOD

(1A) IN TIME DOMAIN:

$$h[n] = T_s\, h_c(nT_s)$$

i.e. SAMPLE THE CONTINUOUS IMPULSE RESPONSE AT T_s.



(1B) IN FREQUENCY DOMAIN:

SOMETIMES WE WANT TO DESIGN THE DIGITAL Filter FROM the laplace transform of the Analog filter, $\mathcal{L}\{h_c(t)\} = H_c(s)$.

DERIVATION of the method!

FIRST!

SUPPOSE $H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$ (PARTIAL FRACTION EXPANSION)

then

$$h_c(t) = \mathcal{L}^{-1}\{H_c(s)\} = \sum_{k=1}^{N} A_k e^{s_k t}$$

then

$$h[n] = T_s h_c(nT_s) = T_s \sum_{k=1}^{N} A_k e^{s_k n T_s}$$

SECOND:

$$\mathcal{Z}\{h[n]\} = \mathcal{Z}\left\{T_s \sum_{k=1}^{N} A_k e^{s_k n T_s}\right\}$$

$$= \sum_{n=-\infty}^{\infty} T_s \sum_{k=1}^{N} A_k e^{s_k n T_s} z^{-n}$$

$$= \sum_{k=1}^{N} A_k T_s \sum_{n=-\infty}^{\infty} \left(e^{s_k T_s} z^{-1}\right)^n$$

$$H(z) = \sum_{k=1}^{N} \frac{A_k T_s}{1 - e^{s_k T_s} z^{-1}}$$

So:

$$\boxed{\text{Given } H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \qquad H(z) = \sum_{k=1}^{N} \frac{A_k T_s}{1 - e^{s_k T_s} z^{-1}}}$$

← FREQ. DOMAIN VERSION of IMPULSE INVARIANCE METHOD.

(2) BILINEAR TRANSFORM

- RECALL, IMPULSE INVARIANCE HAS ALIASING.
- BILINEAR TRANSFORM AVOIDS ALIASING BY "SQUEEZING" $-\infty < \Omega < \infty$ into $-\pi < \omega < \pi$
- It is A FREQUENCY DOMAIN method.

GIVEN ANALOG FILTER $H_c(s)$

$$\boxed{H(z) = H_c(s)\Big|_{s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right) = \frac{2}{T_s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}}$$

USE EITHER FORM.

IMPORTANT ASPECTS OF BILINEAR TRANSform.

- No Aliasing
- Stable analog filter guarantees stable $H(z)$
- Frequency warping

Since the $-\infty \to \infty$ continuous frequency is mapped into $-\pi \to \pi$ discrete-time frequency, something must be "warped" or "squeezed"

Let $s = \sigma + j\Omega$; $z = re^{j\omega}$

then the substitution becomes.

$$s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right)$$

becomes $\sigma + j\Omega = (2/T_s)\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$

After much rearrangement:

$$\sigma = \frac{2}{T_s}\frac{r^2-1}{1+r^2+2r\cos(\omega)} \qquad \Omega = \frac{2}{T_s}\frac{2r\sin(\omega)}{1+r^2+2r\cos(\omega)}$$

Of particular interest is the unit circle in z-plane where $r=1$; then $\sigma = 0$, and

$$\Omega = \frac{2}{T_s}\frac{\sin(\omega)}{1+\cos(\omega)} = \frac{\cancel{2}}{T_s}\frac{\sin(\omega)}{\cancel{2}\cos^2(\frac{\omega}{2})} = \frac{1}{T_s}\frac{\sin(\frac{\omega}{2})\cos(\frac{\omega}{2})+\cos(\frac{\omega}{2})\sin(\frac{\omega}{2})}{\cos^2(\omega/2)}$$

$$= \frac{2}{T_s}\frac{\sin(\omega/2)}{\cos(\omega/2)} = \frac{2}{T_s}\tan\left(\frac{\omega}{2}\right)$$

$$\Omega = \frac{2}{T_s}\tan\left(\frac{\omega}{2}\right)$$ (NOTE! For small ω, $\Omega \approx \omega/T_s$)

And

$$\omega = 2\tan^{-1}\left(\frac{\Omega T_s}{2}\right) ; \quad -\pi < \omega < \pi$$

This is the "warping" found in Bilinear Tx.

(3) Pre-warped Bilinear Transform

By adjusting $H_c(s)$ we can correct for frequency warping at **one** frequency.

Consider an N-th order Butterworth.

$$|H_c(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

where Ω_c is a constant = cutoff frequency in rad/s. If we designed this with Bilinear transform, we know $\Omega = \frac{2}{T_s}\tan(\omega/2)$, so the frequency response of the digital filter is

$$|H(\omega)|^2 = |H_c(\Omega)|^2\Big|_{\Omega = \frac{2}{T_s}\tan\left(\frac{\omega}{2}\right)} = \frac{1}{1 + \left(\frac{\frac{2}{T_s}\tan\left(\frac{\omega}{2}\right)}{\Omega_c}\right)^{2N}}$$

The cutoff frequency occurs at

$$\Omega_c = \frac{2}{T_s}\tan\left(\frac{\omega}{2}\right) \quad \text{or} \quad \omega = 2\tan^{-1}\left(\frac{\Omega_c T_s}{2}\right)$$

when we would have expected it at $\omega = \Omega_c T_s$.

So we pre-warp the analog filter such that:

$$\longrightarrow \quad \Omega_c' = \frac{2}{T_s}\tan\left(\Omega_c T_s/2\right)$$

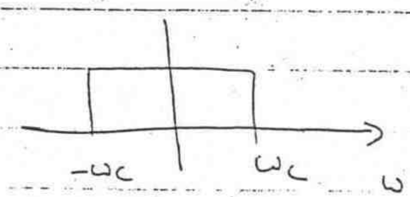
Then.

$$|H(\omega)|^2 = |H_c(\Omega)|^2\Big|_{\Omega = \frac{2}{T_s}\tan\frac{\omega}{2}} = \frac{1}{1 + \left(\frac{\frac{2}{T_s}\tan(\omega/2)}{\frac{2}{T_s}\tan\left(\frac{\Omega_c T_s}{2}\right)}\right)^{2N}}$$

So → Change the analog filter cutoff frequency so the digital filter cutoff is correct ($\omega = \Omega_c T_s$)

FIR Design by Windowing

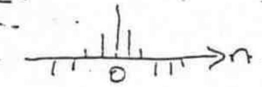
Suppose: you want to approximate an ideal filter (lowpass)



Problem:

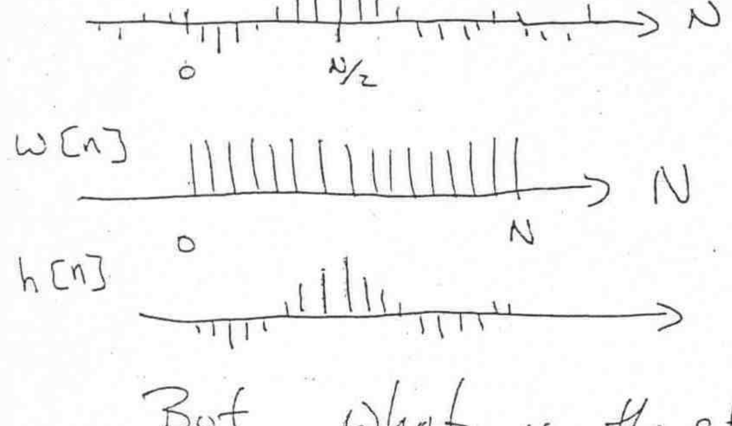
$$h_d[n] = \frac{\sin(\omega_c n)}{\pi n}$$ is noncausal.

↖ desired impulse resp.



So

Assume that you want M-point FIR approximation; now: delay h[n] by M/2 to get "more causal" (same as multiplying by $e^{-j\omega M/2}$ in freq. domain)



One approach to FIR is to simply truncate.

But what is the effect of this truncation in frequency domain?

$$\mathcal{F}\{h_d[n]\,w[n]\} = \mathcal{F}\{h[n]\}$$

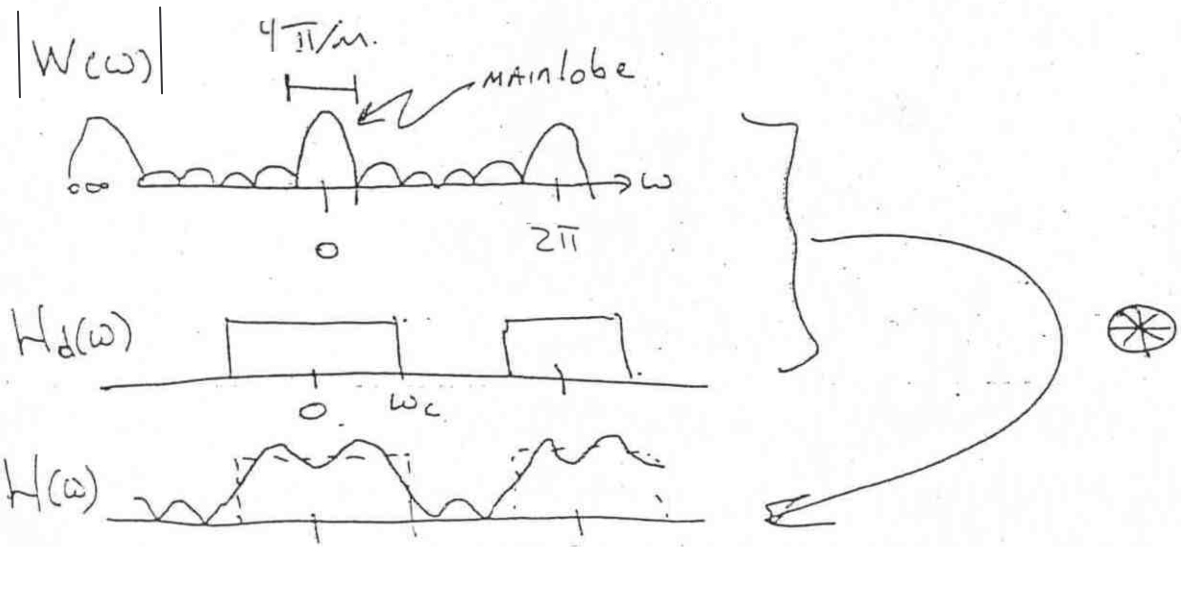
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\alpha)\,W(\omega-\alpha)\,d\alpha.$$

So the resulting filter IS THE CONVOLUTION of the desired response $H_d(\omega)$ with the DTFT OF the window function $w[n]$.

In our case $w[n] = \begin{cases} 1 & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$

So

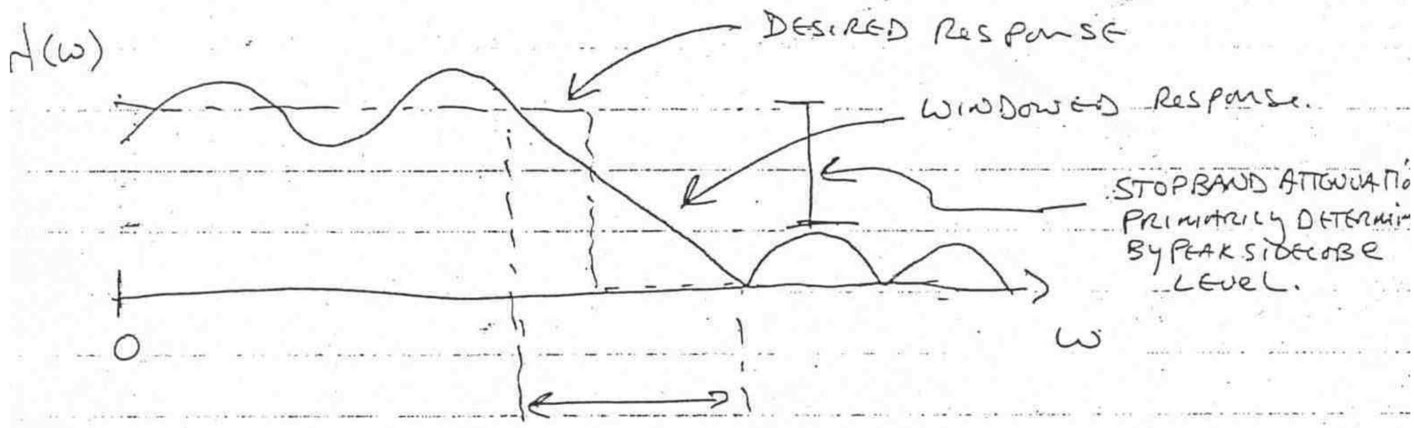
$$W(\omega) = \frac{\sin(\omega M/2)}{\sin(\omega/2)}\, e^{-j\omega(M-1)/2}$$



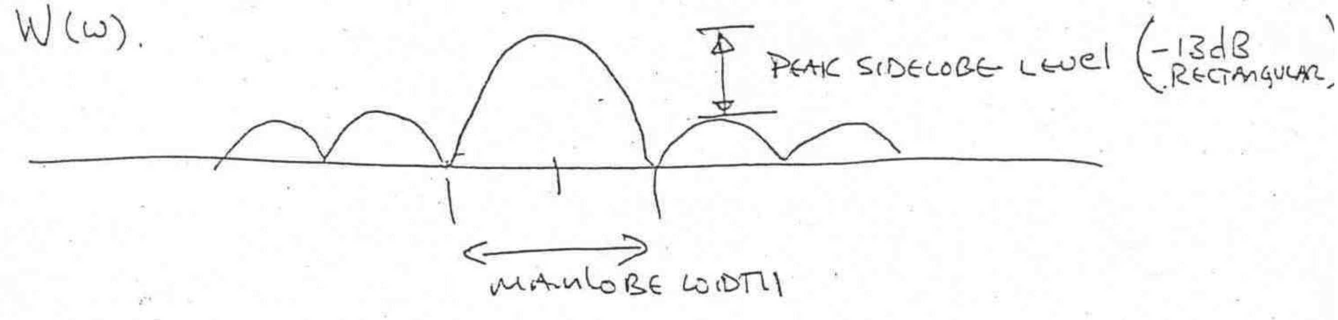
April 8

So!

We can see!



WIDTH OF TRANSITION REGION – PRIMARILY AFFECTED BY MAINLOBE WIDTH.



So

Choice of different window functions LEADS to different SPECTRAL TRADEOFFS.

→ Generally LARGER TRANSITION Region. ⇒ lower "SIDELOBES

PAGE 468 TABLE 7. OPPENHEIM.

OTHER WINDOWS ~~FIGURE 7.29. 7.27~~

FIR FILTER WINDOWS

RECTANGULAR

$$w[n] = \begin{cases} 1 & ; \ 0 \le n \le N_o \\ 0 & ; \ \text{otherwise} \end{cases}$$

Where N_o = window length.

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/N_o & ; \ 0 \le n \le N_o/2 \\ 2 - 2n/N_o & ; \ N_o/2 \le n \le 0 \\ 0 & ; \ \text{otherwise} \end{cases}$$

HANNING

$$w[n] = \begin{cases} \dfrac{1 - \cos(2\pi n/N_o)}{2} & ; \ 0 \le n \le N_o \\ 0 & ; \ \text{otherwise} \end{cases}$$

HAMMING

$$w(n) = \begin{cases} .54 - .46\cos\left(\dfrac{2\pi n}{N_o}\right) & ; \ 0 \le n \le N_o \\ 0 & ; \ \text{otherwise} \end{cases}$$

Blackman

$$w[n] = \begin{cases} .42 - .5\cos\left(\dfrac{2\pi n}{N_o}\right) + .08\cos\left(\dfrac{4\pi n}{N_o}\right) \\ 0 & \text{otherwise} \end{cases}$$

FILTER	PEAK SIDELOBE	MAINLOBE WIDTH
RECT.	-13 dB	$\sim 4\pi/N_o$
BARTLETT	-26 dB	$8\pi/N_o$
HANNING	-30 dB	$8\pi/N_o$
HAMMING	-40 dB	$8\pi/N_o$
BLACKMAN	-57 dB.	$12\pi/N_o$

Filter Architecture

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k'=0}^{M} b_{k'}\, x[n-k']$$

$$\mathcal{Z} \Rightarrow Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k'=0}^{M} b_{k'} z^{-k'}$$

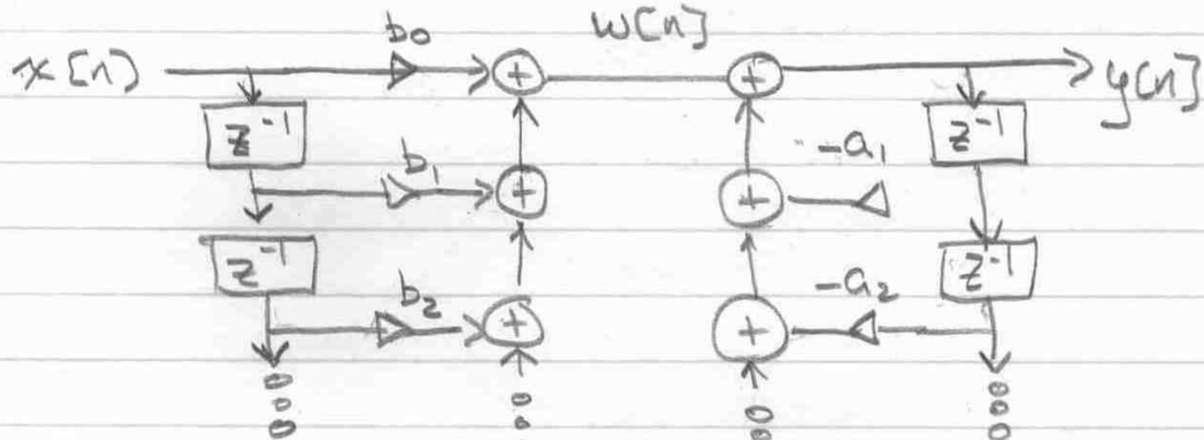
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k'=0}^{M} b_{k'} z^{-k'}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Generally $a_0 = 1$

$$H(z) = \frac{\sum_{k'=0}^{M} b_{k'} z^{-k'}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{N(z)}{D(z)}$$

Rewriting:

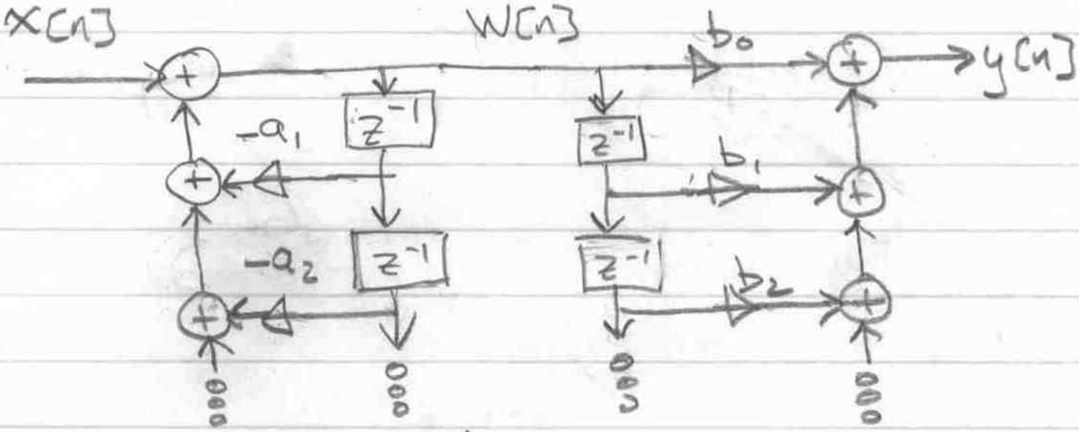
$$y[n] = -a_1 y[n-1] - a_2 y[n-2] \cdots + b_0 x[n] + b_1 x[n-1] \cdots$$
$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) \cdots + b_0 X(z) + b_1 z^{-1} X(z) \cdots$$



$$W(z) = X(z) N(z)$$
$$Y(z) = W(z)/D(z) = X(z) N(z)/D(z)$$

Filter Architecture, other forms

Alternate Architecture

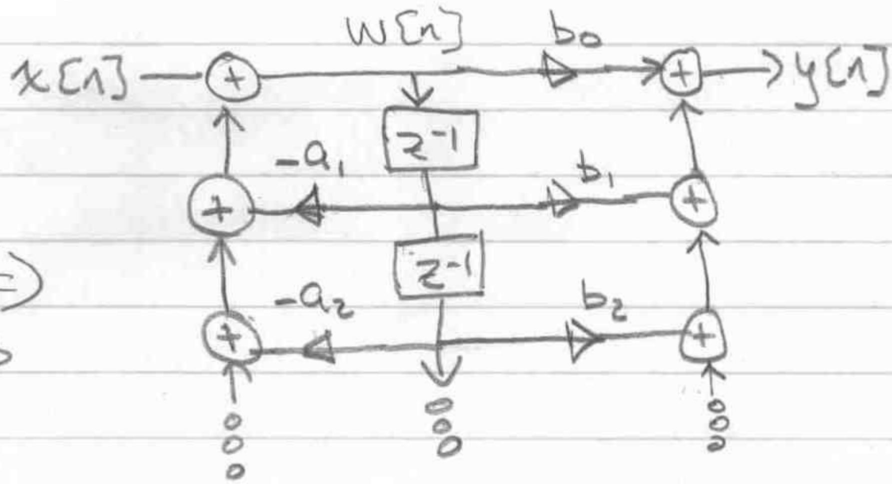


Above $W(z) = X(z)/D(z)$

$$Y(z) = W(z)\,N(z) = X(z)N(z)/D(z)$$

Essentially the above interchanges the order of the $N(z)$ and $1/D(z)$ sections of the filter

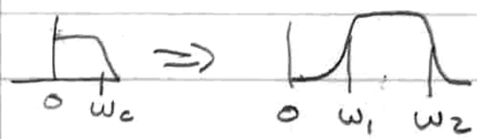
Above Redrawn



USES LESS ⟹ REGISTERS

FILTER TRANSFORMATIONS (SEE PROAKIS & MANOLAKIS)

TYPE	SUBSTITUTION FOR z^{-1}	
LOWPASS TO LOWPASS	$\dfrac{z^{-1}-a}{1-az^{-1}}$	$a = \dfrac{\sin[(\omega_c - \omega_{c\,NEW})/2]}{\sin[(\omega_c + \omega_{c\,NEW})/2]}$
LOWPASS TO HIGHPASS	$-\dfrac{z^{-1}+a}{1+az^{-1}}$	$a = \dfrac{\cos[(\omega_c + \omega_{c\,NEW})/2]}{\cos[(\omega_c - \omega_{c\,NEW})/2]}$
LOWPASS TO BANDPASS	$-\left(\dfrac{z^{-2}-a_1 z^{-1}+a_2}{a_2 z^{-2}-a_1 z^{-1}+1}\right)$	$a_1 = -2\alpha\beta/(\beta+1)$ $a_2 = (\beta-1)/(\beta+1)$



$$\alpha = \frac{\cos[(\omega_1+\omega_2)/2]}{\cos[(\omega_2-\omega_1)/2]}$$

$$\beta = \cot\left(\frac{\omega_2-\omega_1}{2}\right)\tan\left(\frac{\omega_c}{2}\right)$$

LOWPASS TO BANDSTOP	$\dfrac{z^{-2}-a_1 z^{-1}+a_2}{a_2 z^{-1}-a_1 z^{-1}+1}$	$a_1 = -2\alpha/(\beta+1)$ $a_2 = (1-\beta)/(1+\beta)$

Also:

SIMPLE LOWPASS to HIGHPASS TRANSFORMATION:

MULTIPLY $h[n]$ by $(-1)^n$

Misc Stuff

Butterworth Filters

$$|H_c(\Omega)|^2 = \frac{1}{1+\left(s/j\Omega_c\right)^{2N}}$$

- N = order of filter
- $1/2$ power point @ $s = j\Omega_c$

Poles of $|H_c(\Omega)|^2$ AT:

$$\left(s/j\Omega_c\right)^{2N} = -1$$

$$\left(s/\Omega_c\right)^{2N} = -(j)^{2N} = -(-1)^N$$

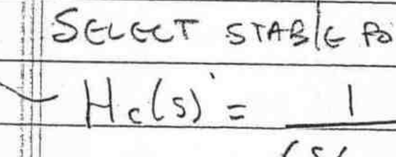
so for N odd, $s/\Omega_c = \sqrt[2N]{1}$

2N roots of 1 on unit circle

N even $s/\Omega_c = \sqrt[2N]{-1}$

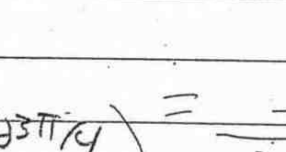
For $N=2$ (2^{nd} order)

$\sqrt[4]{-1}$



For $N=3$

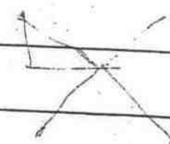
$\sqrt[6]{1}$



Select stable poles (left)

$$H_c(s) = \frac{1}{\left(s/\Omega_c - e^{j3\pi/4}\right)\left(s/\Omega_c - e^{-j3\pi/4}\right)} = \frac{\Omega_c^2}{s^2+\sqrt{2}\,\Omega_c s + \Omega_c^2}$$

2ND ORDER Butterworth



1 Impulse Invariance

PARTIAL FRAC. EXPANSION

$$H_c(s) = \frac{1}{\left(s/\Omega_c - e^{j3\pi/4}\right)\left(\frac{s}{\Omega_c} - e^{-j3\pi/4}\right)}$$

$$= \Omega_c\left(\frac{-j/\sqrt{2}}{s-\Omega_c e^{-j3\pi/4}} + \frac{j/\sqrt{2}}{s-\Omega_c e^{j3\pi/4}}\right)$$

So $H(z) = T_s\Omega_c\left(\frac{-j/\sqrt{2}}{1-e^{\Omega_c T_s e^{-j3\pi/4}}z^{-1}} + \frac{j/\sqrt{2}}{1-e^{\Omega_c T_s e^{-j3\pi/4}}z^{-1}}\right)$

Since $\frac{A_k}{s-s_k} \Rightarrow \frac{A_k T_s}{1-e^{s_k T_s}z^{-1}}$

2 Bilinear

$$H(z) = \left.\frac{\Omega_c^2}{s^2+\sqrt{2}\,\Omega_c s + \Omega_c^2}\right|_{s=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right)}$$

3 Prewarp Bilinear

Suppose $F_s = 8000$ s/s $T_s = 1/8000$ $\Omega_c = 1000\,\text{Hz}\cdot 2\pi$

First Prewarp

$$|H_p(s)|^2 = \frac{1}{1+\left(\frac{s}{j\frac{2}{T_s}\tan\left(\frac{\Omega_c T_s}{2}\right)}\right)^4} = \frac{1}{1+\left(\frac{s}{j(1055)2\pi}\right)^4}$$

So $H_p(s) = \left.\frac{\Omega_c^2}{s^2+\sqrt{2}\,\Omega_c s + \Omega_c^2}\right|_{\Omega_c = (1055)2\pi}$

$$H(z) = \left.H_p(s)\right|_{s=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right)}.$$

Chebyshev Filters

$$|H(s)|^2 = \frac{1}{1+\epsilon^2 C_n^2(s/j\Omega_c)}$$

ϵ^2 SETS PASSBAND RIPPLE

PASSBAND VARIES FROM $1/(1+\epsilon^2)$ to 1

$C_n(\cdot)$ IS DEFINED RECURSIVELY

$$C_n(x) = 2x\,C_{n-1}(x) - C_{n-2}(x)$$

where

$$C_0(x) = 1$$
$$C_1(x) = x$$
$$C_2(x) = 2x^2 - 1$$
$$C_3(x) = 4x^3 - 3x$$
$$C_4(x) = 8x^4 - 8x^2 + 1$$

Notes

1. $C_n(1) = 1$ FOR ALL $C_n(\,)$
2. $|C_n(0)| = 1$ OR 0.