ECGR 6114
Computer Project: Spectral Estimation
Student Name:___________

Part1
Overview

A simple approach to finding the frequency spectrum of a noise-like random signal would be to take the discrete Fourier transform of the signal. However, the spectrum appears quite noisy because the it has a very high variance.

We will employ a number of different methods to estimate the frequency spectrum with reduced variance.

First, let us construct a few random test signals.

Create a random binary (+/-1) signal x, for FIR filter of length Nb, with signal pulsewidth Nb/4, noise q with uniform pdf from -noise/2 to noise/2,

$N := 2^{12}$ $\qquad$ $signal := 1$ $\qquad$ $pulsewidth := 8$

$noise := 2$

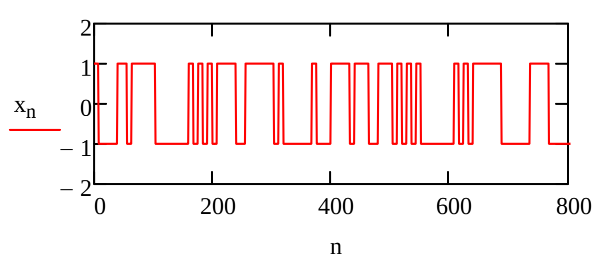
$n := 0 .. N - 1$ $\qquad$ $x_n := signal$ $\qquad$ $pulsewidth = 8$

$pp := 0 .. \mathrm{round}\left(\frac{N}{pulsewidth}\right) + 10$ $\qquad$ $ss_{pp} := \mathrm{rnd}(signal) - \frac{signal}{2}$

$ss := 2 \cdot signal \cdot (\mathrm{rbinom}(\mathrm{length}(x), 1, 0.5) - 0.5)$

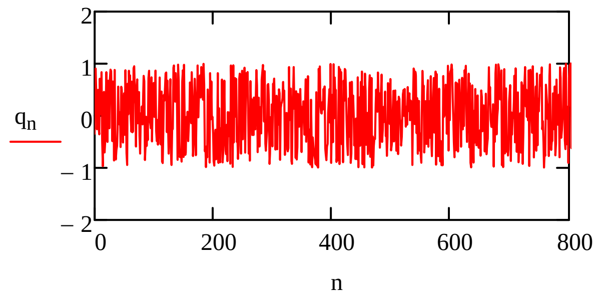
Form the random binary signal x:

$x_n := ss_{\mathrm{trunc}\left(\frac{n}{pulsewidth}\right)}$



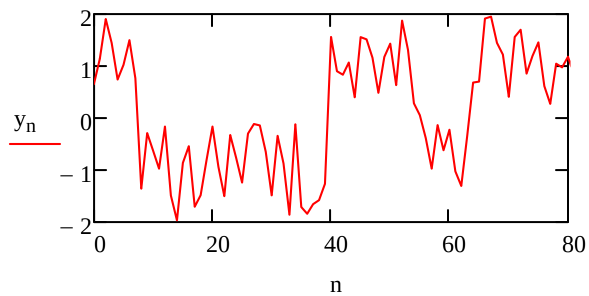
Form uniform pdf noise q :

$$q_n := \text{rnd}(\text{noise}) - \frac{\text{noise}}{2}$$



Form a noisy signal y

$$y_n := x_n + q_n$$



Q1. What is the peak-peak voltage of the random signal x?

Q2. What is the theoretical value of $S_q(w)$, the power spectral density of q ?

Q3. What is the theoretical standard deviation of q?

Part 2
Frequency spectrum of the signals using DFT

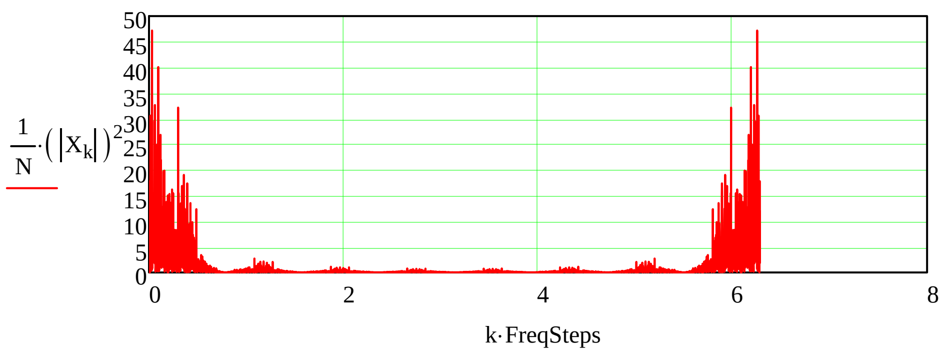
Now, use the Mathcad CFFT function to take the Fast Fourier Transform (FFT) of your sampled data to get the frequency spectrum (this will be a spectral estimate with high variance).
The CFFT is similar to DFT, except Mathcad version differs by a constant factor of npts.
(To see definition of CFFT, see the Mathcad help page for CFFT).

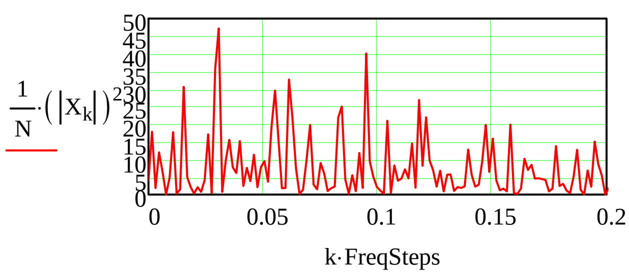
$Ts := 1$ $\quad X := \text{rows}(x)\cdot\text{CFFT}(x)$ $\quad \text{rows}(X) = 4.096 \times 10^{3}$

$\text{FreqSteps} := \frac{\pi\cdot 2}{N\cdot Ts}$ $\quad \text{FreqSteps} = 1.534 \times 10^{-3}$ FreqSteps is frequency stepsize in FFT in rad/s.

$k := 0 .. \text{rows}(x) - 1$ $\quad N = 4.096 \times 10^{3}$

Recall, periodigram defined as (1/N)|DFT{x[n]}|^2 $\quad \left|X_{10}\right| = 353.755$





zoom in on a region

Part 3

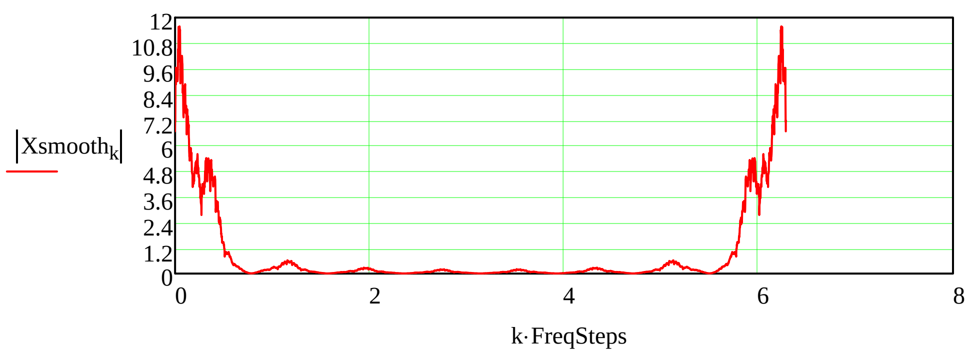
Convolution-Smoothed DFTspectrum (Blackman-Tukey)

We can frm a smoothed spectrum (BlackmanTukey Periodigram) by either multiplying the autocorrelation by a window and taking DFT or by convolving with a window in the frequency domain. Since we have the spectrum from the preceeding section, we will **convolve the window** with the magnitude squared of the spectrum to implement the Blackman Tukey Periodigram below

$$
\mathrm{MagSqSmooth}(x,p) := \left|\begin{array}{l}
sx \leftarrow \mathrm{last}(x) \\
\text{for } \; xx \in 0 .. \mathrm{last}(x) \\
\quad y_{xx} \leftarrow 0 \\
\text{for } \; nn \in 0 .. \mathrm{last}(x) \\
\quad \left| \begin{array}{l}
y_{nn} \leftarrow \displaystyle\sum_{k=-\left(p-\mathrm{trunc}\left(\frac{p}{2}\right)-1\right)}^{\mathrm{trunc}\left(\frac{p}{2}\right)} \left(\left|x_{\mathrm{mod}(\mathrm{length}(x)+nn+k,\,\mathrm{length}(x))}\right|\right)^2 \\
y_{nn} \leftarrow \dfrac{y_{nn}}{p\;\mathrm{rows}(y)}
\end{array}\right. \\
\text{return } \; y
\end{array}\right.
$$

$$\mathrm{Smoo} := 32$$

$$\mathrm{Xsmooth} := \mathrm{MagSqSmooth}(X, \mathrm{Smoo})$$



Q4. What is the shape of the frequency domain window used above in the periodigram smoothing?   triangle/rectangle/sinc/cos?
 (Note that this is not a proper window, see paragraph after 8.70 on p.421)

Q5. What is the theoretical reduction factor in the variance for the above window?

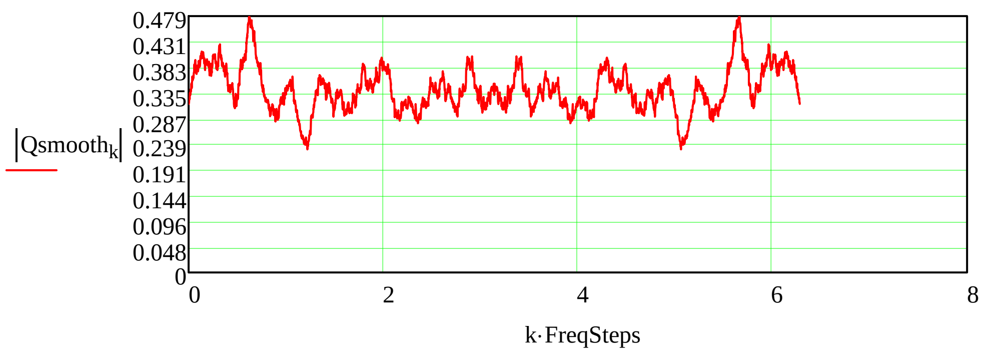
As another example,
apply the smoothing to the spectrum of the noise signal q from part 1

$Q := rows(q) \cdot CFFT(q)$ $Q_0 = 92.782$

$Qsmooth := MagSqSmooth(Q, 100)$

For a moment, temporarily change the smoothing factor to 100 to 1, to see the noise spectrum without smoothing. Dont forget to change it back to 100.

$qsmmax := max\left[\overrightarrow{(|Qsmooth|)}\right]$ $qsmmax = 0.479$



Q6. For q, with unifom pdf, what is the theoretical variance of the noise q ?
(Hint: look at the plot of q in part 1)

Q7. Does the magnitude of the smoothed periodigram power spectrum of q above equal the variance of the noise (see Hayes Example 8.2.1)?

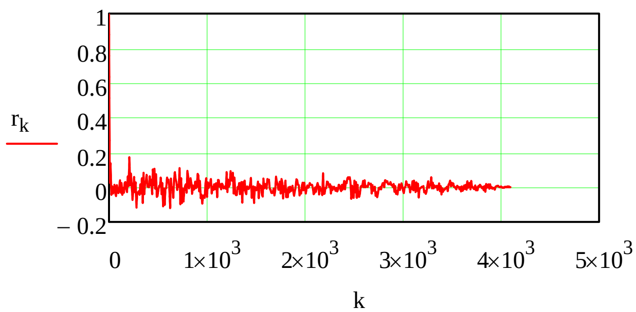
Part 4
Computing the autocorrelation

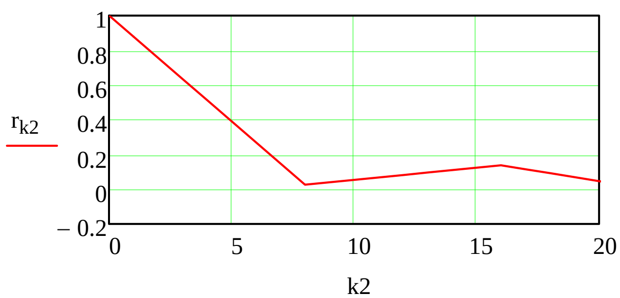
Next, compute the autocorrelation (see eq. 8.4)

$$k2 := 0 .. \text{rows}(x) - 1$$

$$r_{k2} := \frac{1}{\text{rows}(x)} \cdot \sum_{n=0}^{\text{rows}(x)-1-k2} \left(x_{n+k2} \, x_n \right)$$

$r_0 = 1$





Plot the first few points

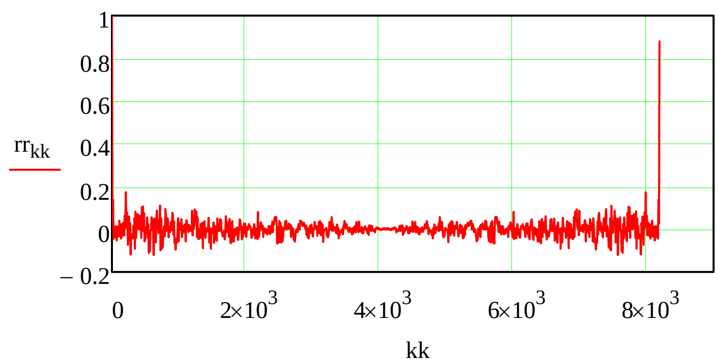
But r[-n]=r[n]
so we must adjust r[n] to include r[-n] using/assuming wraparound for negative n

$kk := 1 .. \text{rows}(r) - 1$ $\quad rr_{2 \cdot \text{rows}(r) - kk} := r_{kk}$ $\quad rr_k := r_k$

$rr_1 = 0.878$ $\quad rr_{\text{rows}(rr)-1} = 0.878$ $\quad kk := 0 .. \text{rows}(rr) - 1$

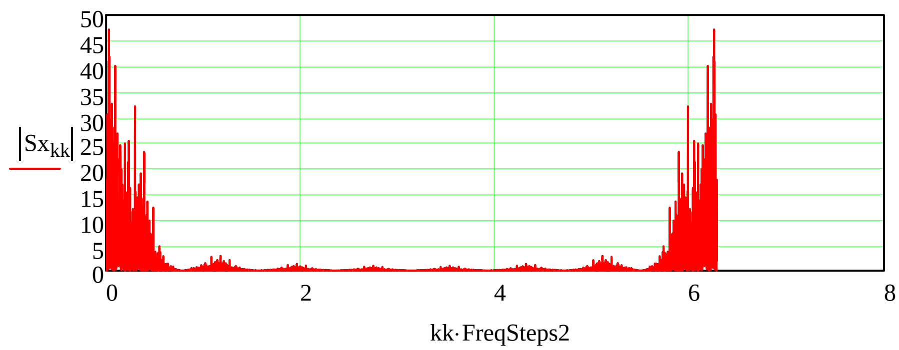
$\text{rows}(r) = 4.096 \times 10^3$ $\quad \text{rows}(rr) = 8.192 \times 10^3$

Finally, the adjusted autocorrelation including x[n] for negative values of (using wraparound)



$$Sx := rows(rr) \cdot CFFT(rr) \qquad kk := 0 .. rows(Sx) - 1 \qquad FreqSteps2 := \frac{\pi \cdot 2}{rows(Sx) \cdot Ts}$$

Finally, the periodigram computed as taking the DFT of the autocorrelation



Q8. How does the above plot of |Sx| result compare with the DFT result for $|X|^2/N$ plotted in Part 2?

differs greatly or quite similar?

Part 5

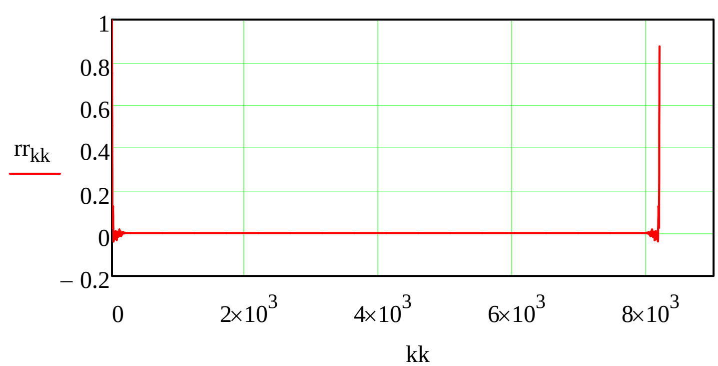
R-multiplied-Smoothed DFTspectrum (Blackman-Tukey)

We can form a smoothed spectrum (BlackmanTukey Periodigram) by either multiplying the autocorrelation by a window and taking DFT or by convolving with a window in the frequency domain. Since we have the autocorrelation from the preceeding section, we will **multiply the autocorrelation** by the window before taking the DFT to compute the spectrum to obtain the Blackman Tukey Periodigram below

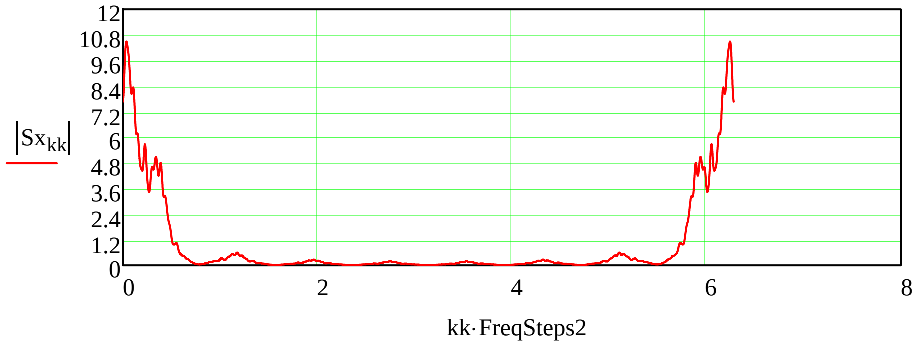
```
rwindow(r,p) := | sx ← last(x)
                | for  xx ∈ 0 .. last(r)
                |   win_xx ← 0
                | for  xx ∈ 0 .. p
                |   win_xx ← (1 − xx/p)
                | for  xx ∈ 1 .. p
                |   win_(rows(r)−xx) ← (1 − xx/p)
                | for  xx ∈ 0 .. rows(r) − 1
                |   y_xx ← r_xx · win_xx
                | return  y
```

$rr := rwindow(rr, 200)$

Finally, the adjusted autocorrelation including x[n] for negative values of (using wraparound)



$Sx := rows(rr) \cdot CFFT(rr)$ $kk := 0 .. rows(Sx) - 1$ $FreqSteps2 := \frac{\pi \cdot 2}{rows(Sx) \cdot Ts}$



Q9. What was the shape of the time domain window used above in the periodigram smoothing?

triangle/rectangle/sinc/cos?

Q10. What was the width of the time domain window used above in the periodigram smoothing? (hint: width includes from -n to n)

Q11. How does the above result compare with the smoothed periodigram Xsmooth in Part 3?

differs greatly or quite similar?

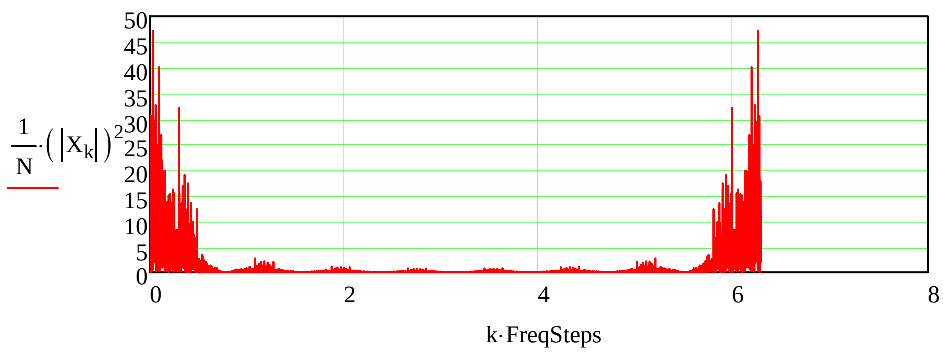
Part 6
Periodigram averaging

Finally,
and use the periodigram averaging method,
breaking the data into λ sections and averaging

$$\text{MagAvg}(x,\lambda) := \begin{array}{|l} \text{siz} \leftarrow \dfrac{\text{rows}(x)}{\lambda} \\ X_{\text{siz}-1} \leftarrow 0 \\ X \leftarrow X\cdot 0 \\ \text{for } \ \text{nn} \in 0..\lambda - 1 \\ \quad \begin{array}{|l} \text{xx} \leftarrow \text{submatrix}(x, \text{nn}\cdot\text{siz}, \text{nn}\cdot\text{siz} + \text{siz} - 1, 0, 0) \\ \text{XX} \leftarrow \text{rows}(\text{xx})\cdot\text{CFFT}(\text{xx}) \\ X \leftarrow X + \dfrac{1}{\text{rows}(\text{XX})}\cdot\overrightarrow{\left[\left(|\text{XX}|\right)^2\right]} \end{array} \\ \text{return } \dfrac{X}{\lambda} \end{array}$$

Recall, periodigram defined as (1/N)|DFT{x[n]}|^2

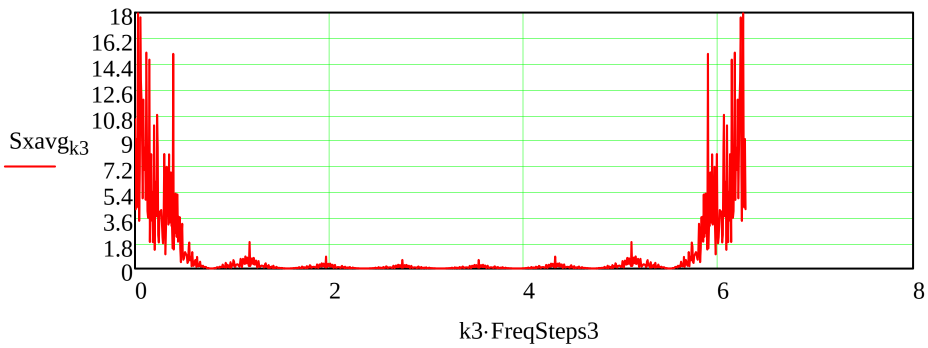
$$\left|X_{10}\right| = 353.755$$



Now, compute the periodigram with the averaging method

$$Sxavg := MagAvg(x, 4)$$

$$k3 := 0 .. rows(Sxavg) - 1 \qquad FreqSteps3 := \frac{\pi \cdot 2}{rows(Sxavg) \cdot Ts}$$



Q12. Change the averaging to average 16 spectra (=6), and does this reduce the variance in the observed spectrum ?
yes/no

Q13. When the averaging equals 16, by what factor (ratio) does the variance theoretically decrease?

Q14. How does the above result compare with the smoothed periodigram Xsmooth in Part 3?

differs greatly or quite similar?