ECGR 6114
Computer Project: FIR filters by windowing
Student Name:___________

Part 1
Continuous-time Butterworth filter

The Butterworth filter is defined by

$$(|H(s)|)^2 := \frac{1}{1 + \left(\frac{s}{i \cdot \Omega c}\right)^{2 \cdot N}}$$

The poles of |H(s)|^2 are at the roorts of the denominator, Dbutter() below

$$\mathrm{Dbutter}(s, \Omega c) := 1 + \left(\frac{s}{i \cdot \Omega c}\right)^{2 \cdot N}$$

To find the roots, first define the filter order, N, and cutoff frequency Ωc in radians/second

$N := 5$ $\quad \Omega c := 1$ $\quad Ts := \frac{1}{\Omega c}$

To use the Mathcad polyroots function,
ceate a vector that contains the coefficients of the polynomial beginning with the constant term and proceeding with increasing powers of s, i.e., a0 + a1 s + a2 s^2.

$k := 0 .. 2 \cdot N$

$\mathrm{DenomCoeffs}_k := 0$ $\quad \mathrm{DenomCoeffs}_0 := 1$ $\quad \mathrm{DenomCoeffs}_{2 \cdot N} := \left(\frac{1}{i \cdot \Omega c}\right)^{2 \cdot N}$

$\mathrm{FilterPoles} := \mathrm{polyroots}(\mathrm{DenomCoeffs})$

These are then the poles of |H(s)|^2.

FilterPoles =

	0
0	-1
1	-0.809-0.5878i
2	-0.809+0.5878i
3	-0.309+0.9511i
4	-0.309-0.9511i
5	0.309+0.9511i
6	0.309-0.9511i
7	0.809-0.5878i
8	0.809+0.5878i
9	1

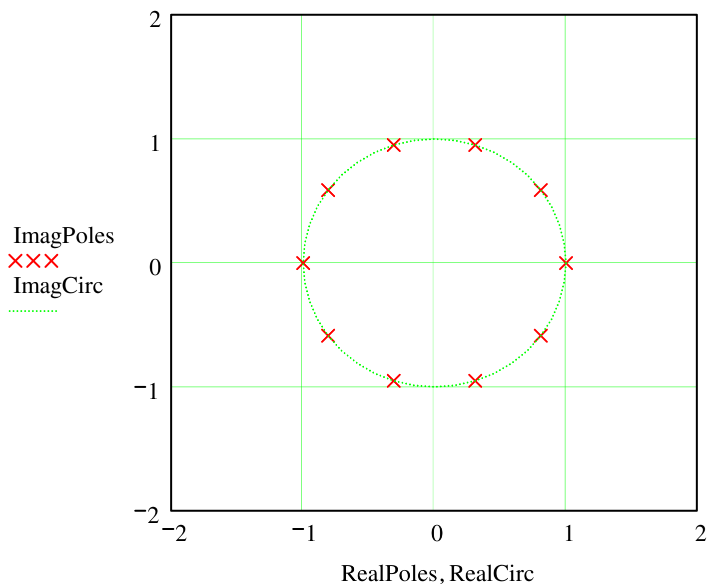
Part 2
Plotting the poles and zeroes of continuous-time filter in the s-plane

and the pole-zero plot

$\text{ImagPoles} := \text{Im}(\text{FilterPoles})$ $\text{RealPoles} := \text{Re}(\text{FilterPoles})$

Lets plot a unit circle too

$un := 0..40$ $\text{ImagCirc}_{un} := \Omega c \cdot \sin\left(2 \cdot \pi \cdot \frac{un}{40}\right)$ $\text{RealCirc}_{un} := \Omega c \cdot \cos\left(2 \cdot \pi \cdot \frac{un}{40}\right)$



Part 3
Finding the stable poles in the s-plane

The stable poles are in the left half plane and gibve rise to H(s) from |H(s)|^2

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StablePoles(vv) := | n ← rows(vv)
                   | cc ← 0
                   | for  nc ∈ 0 .. n − 1
                   |     | xx_cc ← vv_nc   if  Re(vv_nc) < 0
                   |     | cc ← cc + 1   if  Re(vv_nc) < 0
                   | xx
```

StableFilterPoles := StablePoles(FilterPoles)

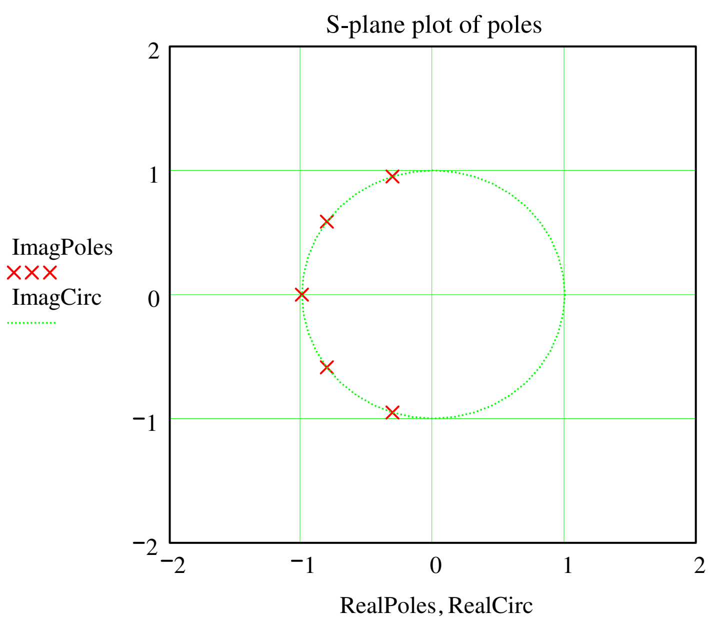
$$\text{StableFilterPoles} = \begin{pmatrix} -1 \\ -0.809 - 0.5878i \\ -0.809 + 0.5878i \\ -0.309 + 0.9511i \\ -0.309 - 0.9511i \end{pmatrix}$$

and the pole-zero plot of the stable poles

$ImagPoles := Im(StableFilterPoles)$ $RealPoles := Re(StableFilterPoles)$

Lets plot a unit circle too

$un := 0..40$ $ImagCirc_{un} := \Omega c \cdot \sin\left(2 \cdot \pi \cdot \frac{un}{40}\right)$ $RealCirc_{un} := \Omega c \cdot \cos\left(2 \cdot \pi \cdot \frac{un}{40}\right)$



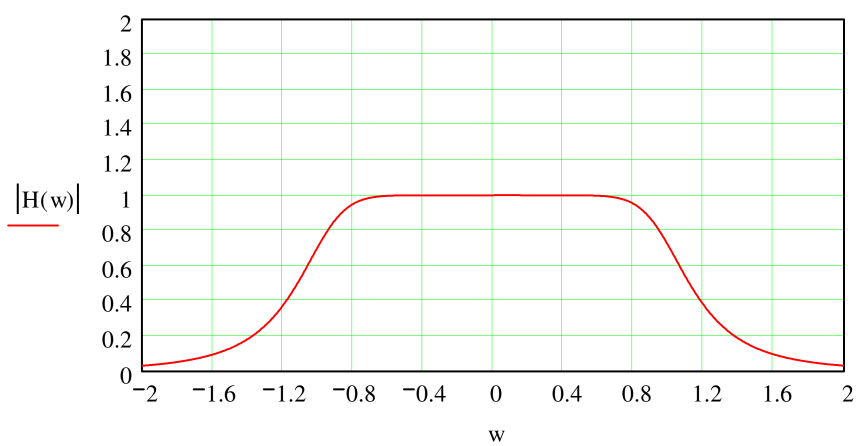
Part 4
The continuous-time filter polynomial in the s-domain

$$\text{ContinuousTimeFilter}(v, s) := \frac{\displaystyle\prod_{n=0}^{\text{rows}(v)-1} \left[(-v)_n\right]}{\displaystyle\prod_{n=0}^{\text{rows}(v)-1} \left(s - v_n\right)}$$

For some reason, the Symbolic expansion below does not work in some versions of Mathcad.

ContinuousTimeFilter(StableFilterPoles, s) complex →

$H(\Omega) := \text{ContinuousTimeFilter}(\text{StableFilterPoles}, i\cdot\Omega)$ $\qquad H(\Omega c) = -0.5 + 0.5i$



Part 5
Impulse response of continuous-time filter

$$H(s) := \text{ContinuousTimeFilter}(\text{StableFilterPoles}, s)$$

Take inverse Laplace transform along the imaginary axis

$$nw := 5$$

$$h(t) := \frac{1}{2\cdot\pi}\cdot\int_{-nw\cdot\Omega c}^{nw\cdot\Omega c} H(i\Omega)\, e^{i\Omega\cdot t}\, d\Omega \qquad \Omega c = 1$$

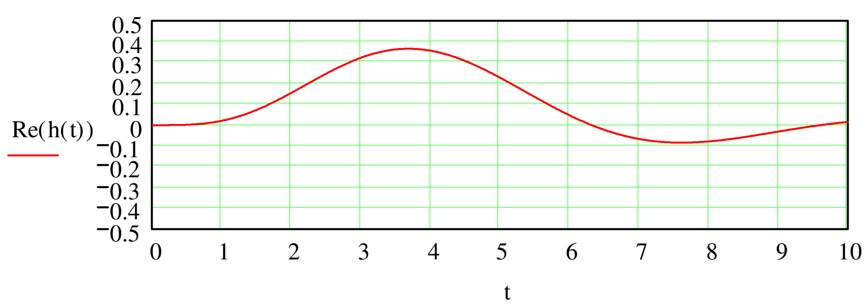
Check against piecewise approximation to integral below

$$npts := 200$$

$$h2(t) := \sum_{k=-npts}^{npts} \frac{H\left(ik\cdot\frac{nw\cdot\Omega c}{npts}\right) e^{i\left(k\cdot\frac{nw\cdot\Omega c}{npts}\right)\cdot t}}{2\pi}\cdot\left(\frac{nw\cdot\Omega c}{npts}\right)$$

$$h(0) = -6.2895\times 10^{-5} - 9.3022i\times 10^{-12} \quad h(1) = 0.021 + 6.1232i\times 10^{-10} \qquad \frac{nw\cdot\Omega c}{npts} = 0.025$$

$$h2(0) = -6.2128\times 10^{-5} - 3.0598i\times 10^{-1} \quad h2(1) = 0.021 + 5.4336i\times 10^{-10}$$



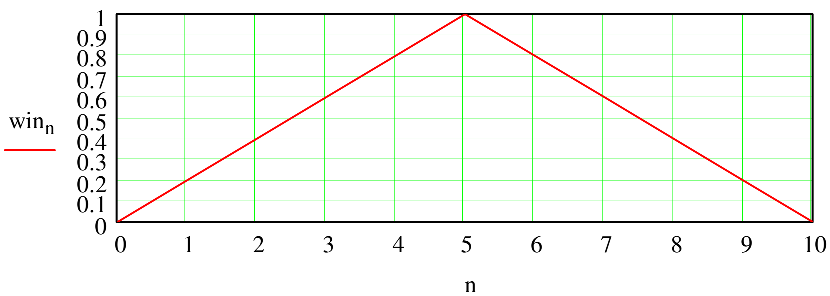
Part 6
Bartlett Windowed filter

The Bartlett window is a triangular window with width of M points.
It has a starting with a value of 0 at n=0,
rising to a value of 1 at n=M/2, and falling back to zero at n=M.
Let the M=point bartlett window win[n] be

$$M := 10$$

$$n := 0..M$$

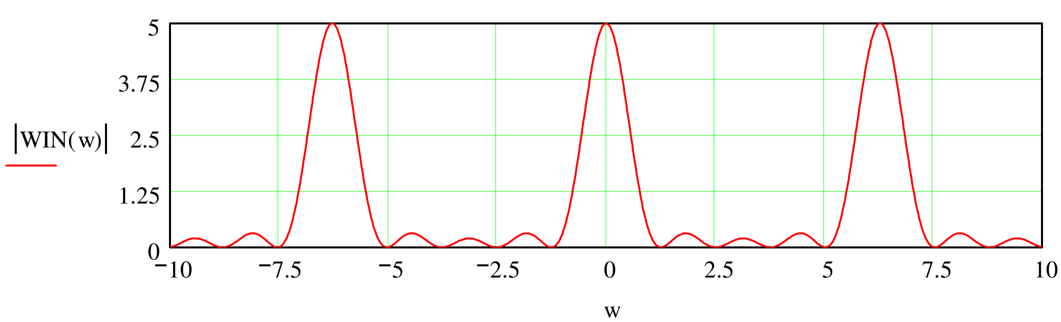
$$win_n := \text{if}\left[\left(n \le \frac{M}{2}\right), 2\frac{n}{M}, 2 - 2\frac{n}{M}\right]$$



The frequency spectrum (DTFT) of the window is

$$WIN(\omega) := \sum_{n=0}^{rows(win)-1} win_n \cdot e^{-i \cdot \omega \cdot n}$$

$WIN(0) = 5$ $\qquad$ $win_1 = 0.2$



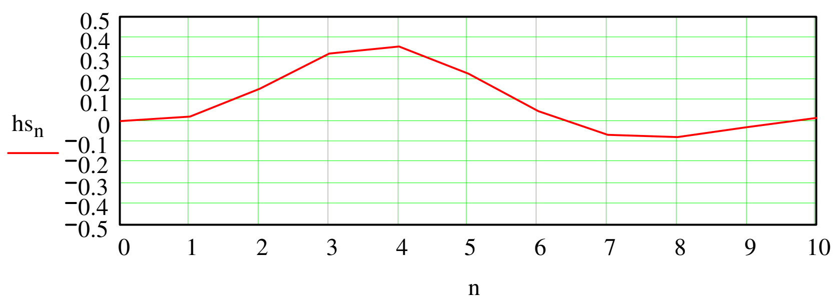
Next, sample the impulse response of the cuninuous-time filter, h(t) to form hs[n]=h(nTs) for the first M points that will comprise the discrete-time filter impulse response.

$Ts = 1 \qquad fs := \frac{1}{Ts} \qquad fs = 1$

$\Omega c = 1 \qquad fc := \frac{\Omega c}{2\pi} \qquad fc = 0.1592$

$\frac{fs}{fc} = 6.2832$

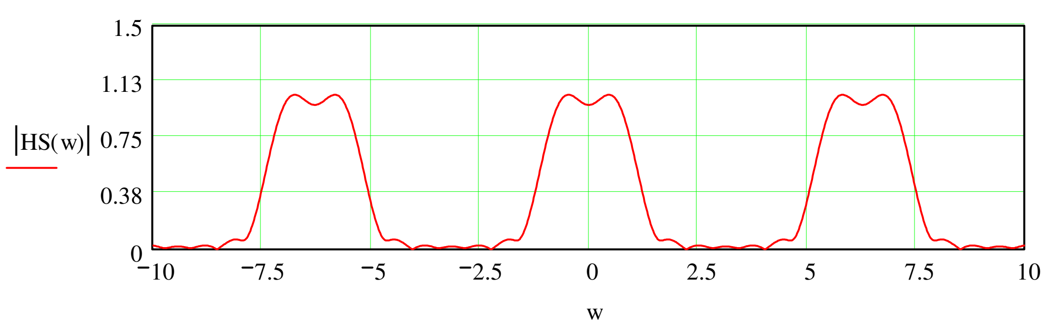
$hs_n := Re(h(n \cdot Ts))$



Now take the DTFT of the sampled signal.
(this corresponds to a rectanglar windowed filter)

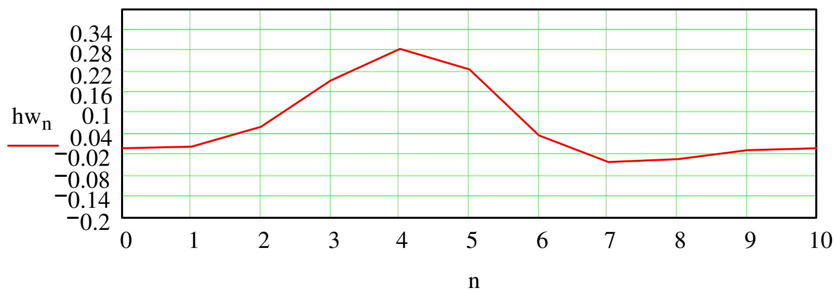
$$HS(\omega) := \sum_{n=0}^{rows(hs)-1} hs_n \cdot e^{-i \cdot \omega \cdot n}$$

$HS(0) = 0.97 \qquad hs_1 = 0.021$



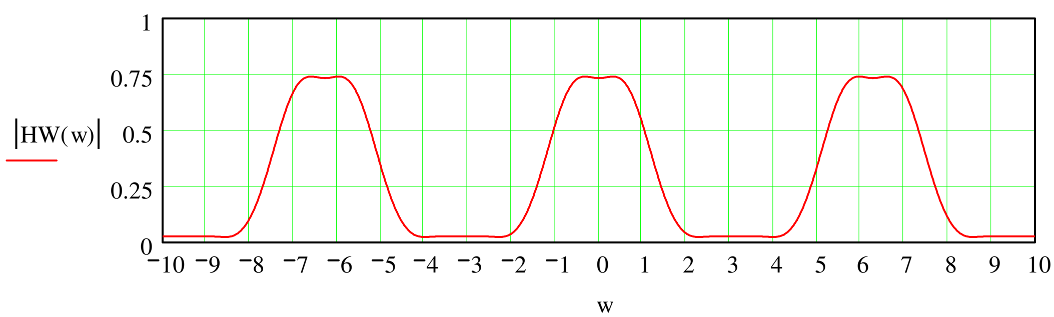
Now form the windowed sampled signal hw[n]=hs[n]win[n]

$$hw_n := win_n \cdot hs_n$$



$$HW(\omega) := \sum_{n=0}^{rows(hw)-1} hw_n \cdot e^{-i \cdot \omega \cdot n}$$

$HW(0) = 0.7328$ $\quad hw_1 = 4.1994 \times 10^{-3}$



Q1. .Which FIR filter has larger passband ripple in the above example plots, Bartlett or rectangular windowed?

Q2. Which FIR filter has a smaller theoretical main lobe width, Bartlett or rectangular windowed?

Q3. Which FIR filter has smaller magnitude of frequncy response at $\omega = 1.5$ rad/sample, Bartlett or rectangular windowed?

Q4. Why is the magnitude of the frequncy response of the rectangular FIR filter at $\omega = 1.5$ rad/sample less than the Bartlett windowed?.

Q5-Q8. Change the window formula above to use an M=10 point Hamming window, and replot :

the window win[n],
window freq response WIN(ω),
widowed impulse response hw[n]
frequency reponse of the windowed filter HW(ω)