ECGR 6114
Computer Project: z-transform
Student Name:___________

Part 1
DFT of windowed sinusoid

The DTFT X(ω) is computed from a discrete-time sampled signal x[n], but has a continuous frequency variable ω. The DFT is sampled in time and frequency, and is equal to the DFT at each frequncy sample, where X[k] = X(ω) at ω = 2πk/N where N is the number of samples in the time and frequency domains. In addition, the time and frequency domains both become periodic, with period N.

The DFT and inverse DFT are defined as

$$X(k) := \sum_{n=0}^{N-1} x(n)\, e^{\frac{(-i\cdot 2\cdot\pi)\cdot n\cdot k}{N}} \qquad x(n) := \left(\frac{1}{N}\right)\cdot\sum_{k=0}^{N-1} X(k)\, e^{\frac{(i\cdot 2\cdot\pi)\cdot n\cdot k}{N}}$$

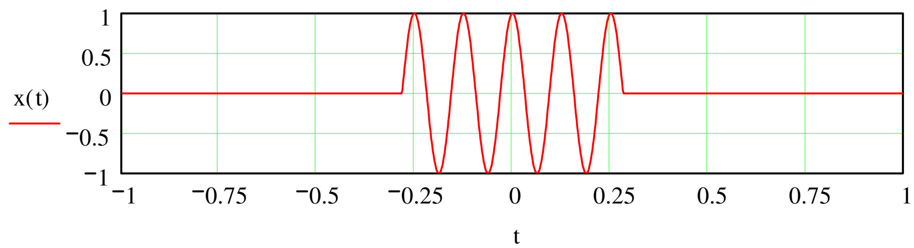
As an example consider a windowed sinusoid

$\text{rect}(t) := \Phi(t + 0.5) - \Phi(t - 0.5)$ Define rect().

$\Omega c := 50$ $\qquad$ $\text{numcycles} := 4.5$

$\tau := \frac{\text{numcycles}\cdot 2\cdot\pi}{\Omega c}$ $\qquad$ $\tau = 0.565$ $\quad$ Set the width τ of the rectangular window, rect(t/τ)

$$x(t) := \cos(\Omega c\, t)\cdot \text{rect}\left(\frac{t}{\tau}\right)$$



To use the Discrete Fourier Transform, the signal must first be sampled.
The sampled signal is defined as xs(n), where n is the time-domain sample index,
where the DFT is Xs(k), k is the frequency-domain sample index,
and indices k and n go from 0 to npts-1, where npts is the number of sample points.
Ts is the sampling time period.
tmax sets the time window over which the sampling is done (-tmax to tmax).

$tmax := 2\cdot\tau$ $\quad tmax = 1.131$ set the sampling window to twice the length of the window

$$Ts := \frac{\left(2\cdot\frac{\pi}{\Omega c}\right)}{64} \qquad Ts = 1.963\times 10^{-3}$$

Set sampling time Ts to be much less than one cycle of the carrier frequency

$\Omega s := 2\cdot\frac{\pi}{Ts}$ $\quad \Omega s = 3.2\times 10^{3}$ $\quad Fs := \frac{1}{Ts}$ $\quad Fs = 509.296$ Sampling frequency=Fs Hertz.

$$npts := 2^{\left(\mathrm{floor}\left(\log\left(\mathrm{floor}\left(\frac{tmax}{Ts}\right),2\right)\right)+1\right)}$$

Set number of points to a power of two. (The equation is a bit cumbersome.)

$npts = 1.024\times 10^{3}$ $\quad npts\cdot Ts = 2.011$

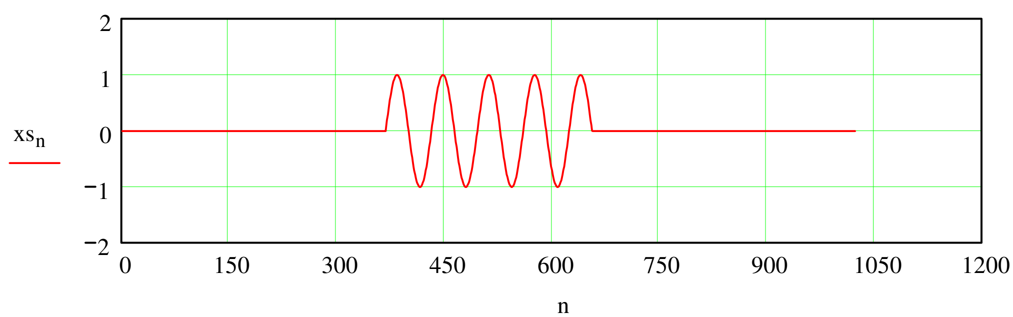
$n := 0 .. npts - 1$

$k := 0 .. npts - 1$

initialize the time and frequency indices, n and k

$$xs_n := x\left(n\cdot Ts - npts\cdot\frac{Ts}{2}\right)$$

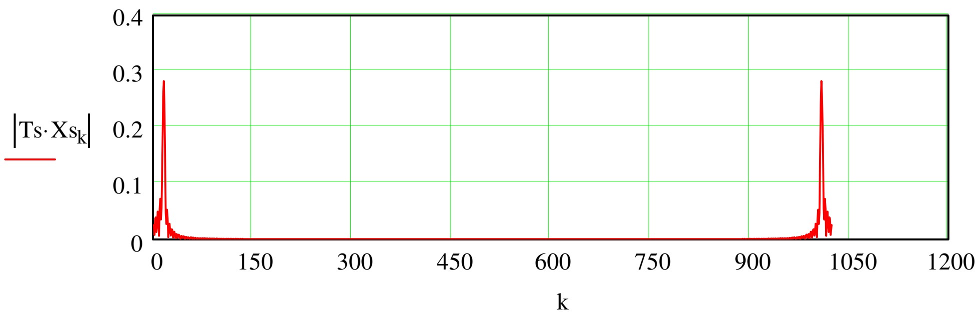
Finally, xs_k is our sampled signal.
It is a vector consisting of the value of x(t) at npts equally spaced time intervals, spaced Ts apart.



Next, compute the DFT using the formula below instead of the Mathcad built-in FFT

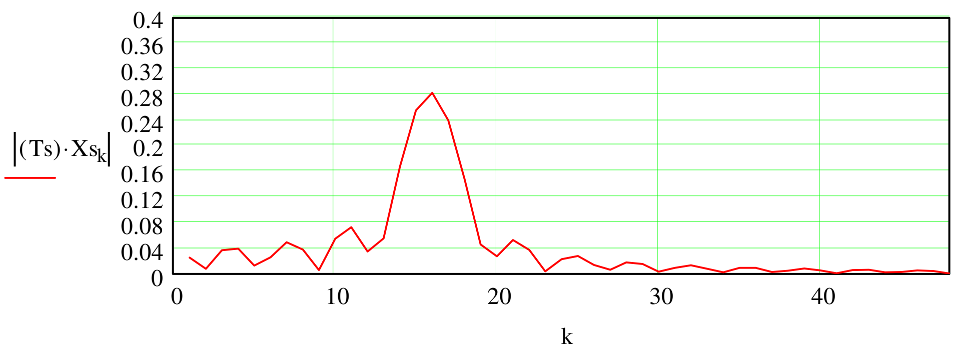
$k := 1 .. \mathrm{rows}(xs) - 1$

$$Xs_k := \sum_{n=0}^{\mathrm{rows}(xs)-1} xs_n\, e^{\frac{(-i\cdot 2\cdot\pi)\cdot n\cdot k}{\mathrm{rows}(xs)}}$$



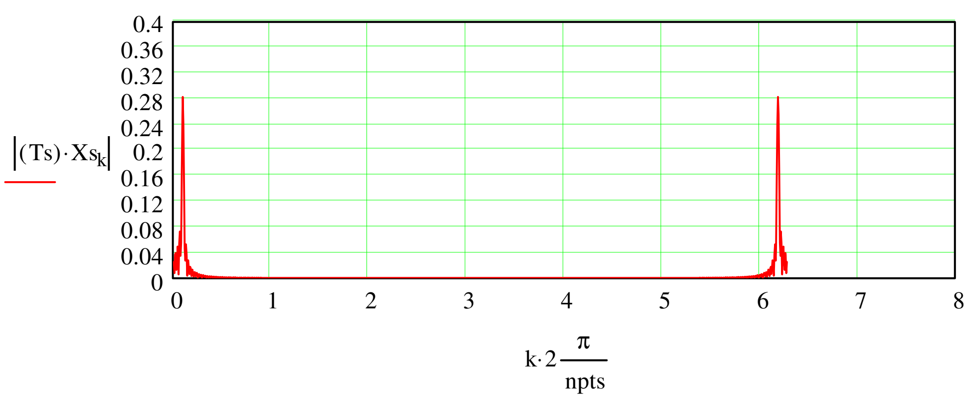
Next, zoom in on the low frequencies:

$nmax := floor\left(3 \cdot \frac{npts \cdot \Omega c}{\Omega s}\right)$ $\quad nmax = 48 \quad$ Lets just plot up to n=nmax, in the plot below



Recall, the frequency index k corresponds to discrete time frequency steps of $\Delta\omega=2\pi/npts$ radians/sample which corresponds to continuous-time frequency steps of $\Delta\Omega=2\pi/(npts*Ts)$ radians/second.

Below, the frequency axis is converted to discrete-time frequency ω and ranges from 0 to 2π radians/sample.



Q1. Change the frequency Ωc to 50 rad/s and replot the signals.

Q2. With Ωc = 50 rad/s, what frequency index k corresponds to the spectral peak of the continuous-timen frequency Ωc = 50 rad/s ?

Q3. With Ωc = 50 rad/s, what is the corresponding discrete-time frequency ω ?

Part 2
Matrix form of DFT

The DFT X[k] can also be expressed and computed in vector-matrix form.
Let X denote the N-point frequency domain column vector, let x denote the N-point time domain column vector, and let F denote the NxN Fourier matrix.
Then, the DFT may be rewritten as

X = F x

As an example consider a 4-point fourier transform,
where the sampled discrete-time data is x = {1, 1, 1, 1}.

$$x := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Define indices m and n for the rows and columns of the matrix F,
and define F from those two indices:

$N := 4 \qquad m := 0 .. N-1 \qquad n := 0 .. N-1$

$$F_{m,n} := e^{\frac{(-i \cdot 2 \cdot \pi) \cdot m \cdot n}{N}} \qquad F = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

Finally, compute the DFT

$$X := F \cdot x \qquad X = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that the result is a DC spectrum,
consisting only of X[0]=1,
with all other frequencies X[1]=X[2]=X[3]=0.

The inverse Fourier transform is then another matrix equal to the inverse of F:

$$IF := F^{-1} \qquad IF = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25i & -0.25 & -0.25i \\ 0.25 & -0.25 & 0.25 & -0.25 \\ 0.25 & -0.25i & -0.25 & 0.25i \end{pmatrix}$$

Checking the inverse Fourier matrix:

$$x := IF \cdot X \qquad x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Q4. Change the original time domain signal to x = {1, -1, 1, -1} and recompute the DFT above to get the new value for X. Why is the new DC value now zero?

Q5. How many complex multiplications (assume all multiplications are complex) does it take to compute the product of the fourier matrix F times the time vector x?

Part 3
The z-transform

The 2-sided z transform is defined for the discrete time sequence
x[n] as X(z) given below

$$X(z) := \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n}$$

Note: the 2-sided z-transform is incomplete unless accompanied by the ROC (region of convergence) specification.

For finite length nonzero sequence, the limits become finite.
If x(n) is a vector in Mathcad holding the samples, the equation is rewritten as:

$$X(z) := \sum_{n=0}^{rows(x)-1} x_n\cdot z^{-n}$$

The inverse z-transform is the following closed contour integral in the region of convergence of the z-plane :

$$x(n) := \left(\frac{1}{j\cdot 2\cdot \pi}\right)\cdot \oint X(z)\, z^{n-1}\, dz$$

The DTFT is related to the z-transform by

$$X(\omega) = X(z) \text{ at } z = e^{j\omega}$$

Hence, the DTFT is z-transform evaluated on the unit circle.
So, the DC response is at z-1, and the highest frequency of $\omega = \pi$ radian/sample corresponds to z = -1.

Part 4
Mathcad Built-in z-transform

First,
use the Mathcad z-transform function to take the z-transform of your sampled data.
The function must be represented as a function of the time variable, n.
Note: it appears that the built-in z-transform is a one-sided z-transform,
and ssumes thet the function equals zero for n < 0.

$$0.5^{-n} \text{ ztrans}, n \;\rightarrow \frac{z}{(z-2.)}$$

$$\frac{z}{(z-2.)} \text{ invztrans}, z \;\rightarrow 1.0000000000000000000 \cdot 2.0000000000000000000^{n}$$

$$1 \text{ ztrans}, n \;\rightarrow \frac{z}{(z-1)}$$

This z-transform result shows that the function is the one-sided z-transform since Z{ u[n] } = z/(z-1); |z|>1

Quite often the function does not seen to work well,
in some editions of Mathcad the following z-transform results
in the un-resolved restult of 'ztrans(δ (n,1),n,z)'
as seen below:

$$\delta(n,1) \text{ ztrans}, n \;\rightarrow \text{ztrans}(\delta(n,1), n, z)$$

Note: δ(n,m) is the Kronecker delta function,

δ(n,m) = 1 if n=m, 0 otherwise

$$\delta(1,0) = 0$$

$$\delta(0,0) = 1$$

Q6. Use the heaviside function Φ to express the delayed unit step u[n-2]
and use it to replace the "3" in the z-transform equation below
and find the z-transform of u[n-3] symbolically.

$$3 \text{ ztrans}, n \;\rightarrow 3 \cdot \frac{z}{(z-1)}$$

Part 5
Another way to do z-transform

Instead of using the Mathcad built-in z-transform,
use the following method for finite sequences.

Suppose the sequence is causal and finite, and can be represented as a vector x,
then the z-transform becomes:

$$Zv(x,z) := \sum_{n=0}^{rows(x)-1} x_n z^{-n}$$

An example:

$$x := \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \qquad Zv(x,z) \rightarrow 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3}$$

and another example

$$y := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 4 \end{pmatrix} \qquad Zv(y,z) \rightarrow 1 + \frac{4}{z^3}$$

Q7. Change the vector x to represent the function $(-2)^{-n}$ u[n] - $(-2)^{-n}$ u[n-5]
to find the z-transform symbolically.

Part 6
Inverse z-transform

Instead of using the Mathcad built-in inverse z-transform,
use the following method.

Most commonly, inverse z-transform of closed form type is done by a look-up table, just as for Laplace transform.

The following method is a numerical implementstion of the inverse z-transform integral

$$x_n = \int_{closedContour} X(z)\, z^{n-1}\, dz$$

where xn is x[n].

To simplify the integral a bit, let us choose a contour as a circle of radius r in the z-plane, and rotate the angle ω from 0 to 2π to follow the circular contour.
Make sure that r is in the region of convergence!

$$invZ(f, x, r, n) := \int_0^{2\pi} f\left(x, r\cdot e^{-i\cdot\omega}\right)\cdot\left(r\cdot e^{-i\cdot\omega}\right)^{n-1}\cdot\frac{\left(r\cdot e^{-i\cdot\omega}\right)}{2\cdot\pi}\, d\omega$$

Repeating the example from before:

$$x := \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \qquad Zv(x,z) \rightarrow 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3}$$

and checking out z-trsnsform at a radius of 10.
(assuming a stable system, as is obvious since the sequence is finite length)

$$invZ(Zv, x, 10, 0) = 1$$

$$invZ(Zv, x, 10, 1) = 2$$

$$invZ(Zv, x, 10, 2) = 3$$

$$invZ(Zv, x, 10, 3) = 4$$

$$invZ(Zv, x, 10, 4) = -8.482\times 10^{-13} + 3.979i\times 10^{-13}$$

Q8. Change the vector x to {1,0,1,1}, and state whether the inverse z-transform results at the left are still correct?.

Part 7
Plotting the z-transform

$$a := \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \qquad b := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$A(z) := z^{(\mathrm{rows}(a)-1)} \cdot Zv(a,z) \text{ expand}, z \rightarrow z^2 + \frac{1}{2} \cdot z + \frac{1}{2}$$

$$B(z) := z^{(\mathrm{rows}(a)-1)}\ Zv(b,z) \text{ expand}, z \rightarrow z^2 + 2 \cdot z + 3$$

$$H(z) := \frac{B(z)}{A(z)} \qquad B(z) \text{ expand}, z \rightarrow z^2 + 2 \cdot z + 3$$

In some editions of Mathcad, the symbolic expansion of H(z) will not evaluate properly below if the elements of a and b are decimal (i.e., 0.5) instead of integer rational fractions (i.e., 1/2).

$$H(z) \rightarrow \frac{\left(z^2 + 2 \cdot z + 3\right)}{\left(z^2 + \frac{1}{2} \cdot z + \frac{1}{2}\right)} \qquad H(1) = 3$$

Before plotting, prevent divide by zero and set poles to "maxplot" height

$$maxPlot := 10$$

$$Amag(rez, imz) := \mathrm{if}\left[\left(|rez + i \cdot imz| < \frac{1}{maxPlot}\right), maxPlot, |A(rez + i \cdot imz)|\right]$$

$$Bmag(rez, imz) := \mathrm{if}\left[\left(|rez + i \cdot imz| < \frac{1}{maxPlot}\right), maxPlot, |B(rez + i \cdot imz)|\right]$$

$$Hmag1(rez, imz) := \mathrm{if}\left[\left(|A(rez + i \cdot imz)| < \frac{1}{maxPlot}\right), maxPlot, \left|\frac{|B(rez + i \cdot imz)|}{|A(rez + i \cdot imz)|}\right|\right]$$

$$Hmag(rez, imz) := \mathrm{if}\left[(Hmag1(rez, imz) > maxPlot), maxPlot, Hmag1(rez, imz)\right]$$

$$Hmag(1,0) = 3 \qquad Hmag(0,0) = 6$$

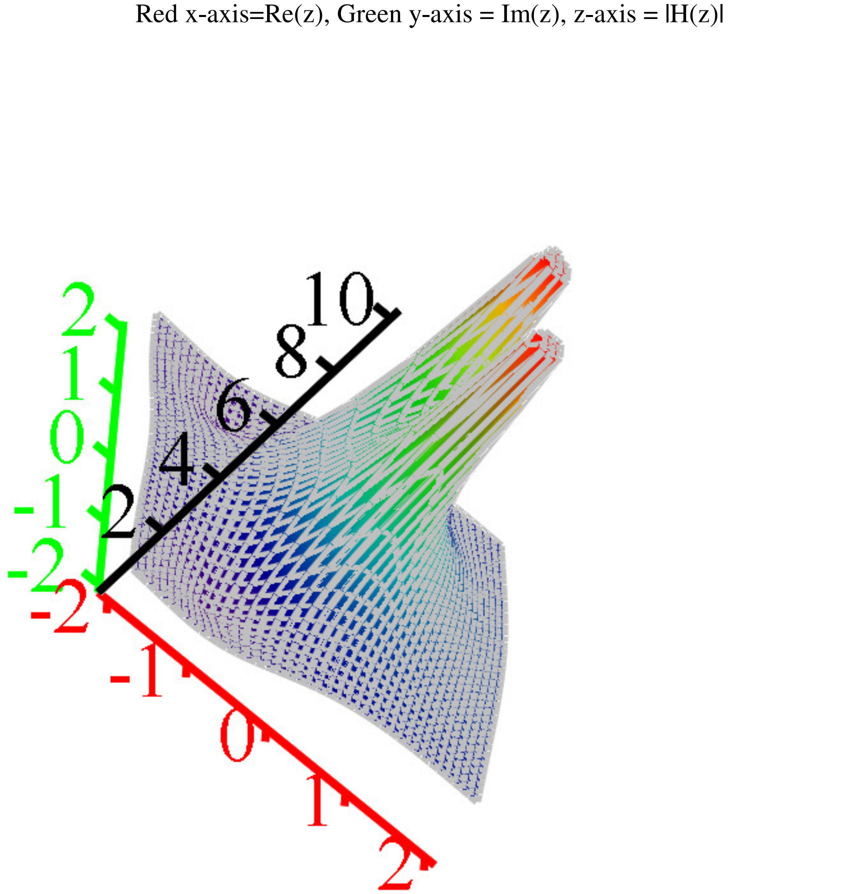
Poles and Zeroes of H(z):

$$Poles := \frac{1}{\text{polyroots}(a)} \qquad Zeroes := \frac{1}{\text{polyroots}(b)}$$

$$Poles = \begin{pmatrix} -0.25 + 0.661i \\ -0.25 - 0.661i \end{pmatrix} \qquad Zeroes = \begin{pmatrix} -1 + 1.414i \\ -1 - 1.414i \end{pmatrix}$$

Click on the plot to rotate it,
double-click to experiment with settings
Red x-xis=Re(z), Green y-axis = Im(z), z-axis = |H(z)|

Red x-axis=Re(z), Green y-axis = Im(z), z-axis = |H(z)|



Hmag

Q9. Redefine A(z) and B(z) in part 5 so that the poles are at z = 0.5 +/- j 0.5 and so the zeroes are at z = +/- 1 and replot the results above.

Part 8
Plotting the poles and zeroes

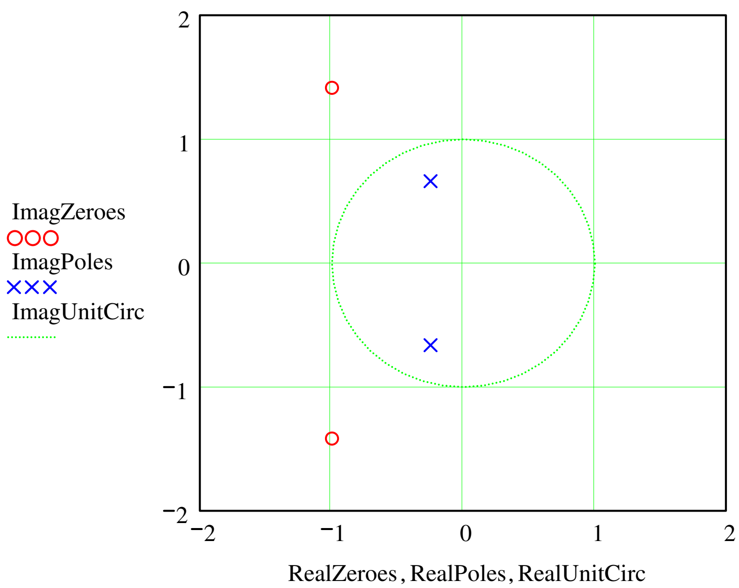
and the pole-zero plot

$\mathrm{ImagZeroes} := \mathrm{Im}(\mathrm{Zeroes})$ $\mathrm{RealZeroes} := \mathrm{Re}(\mathrm{Zeroes})$

$\mathrm{ImagPoles} := \mathrm{Im}(\mathrm{Poles})$ $\mathrm{RealPoles} := \mathrm{Re}(\mathrm{Poles})$

Lets plot a unit circle too

$un := 0..40$ $\mathrm{ImagUnitCirc}_{un} := \sin\left(2\cdot\pi\cdot\frac{un}{40}\right)$ $\mathrm{RealUnitCirc}_{un} := \cos\left(2\cdot\pi\cdot\frac{un}{40}\right)$



Q10. Are the poles and zeroes in the plot above in the correct locations after redefining A(z) and B(z) in Q9 above?

Part 9
Left and Right-sided inverse z-transform solutions

$$H(z) \rightarrow \frac{\left(z^2 + 2\cdot z + 3\right)}{\left(z^2 + \frac{1}{2}\cdot z + \frac{1}{2}\right)}$$

Define a new inverse z-transform, since the above H(z) does not have an argument (i.e.,H(x,z)) as was required in the firs inverse z-transform we defined

$$\text{invZ2}(f, r, n) := \int_0^{2\pi} f\left(r\cdot e^{-i\cdot\omega}\right)\cdot\left(r\cdot e^{-i\cdot\omega}\right)^{n-1}\cdot\frac{\left(r\cdot e^{-i\cdot\omega}\right)}{2\cdot\pi}\, d\omega$$

If the first form does not converge, use the approximation below.

$$nw := 80 \qquad dw := \frac{\pi}{nw}$$

$$\text{invZ3}(f, r, n) := \sum_{p=-nw}^{nw-1} f\left[r\cdot e^{i\cdot\left(p\cdot dw+\frac{dw}{2}\right)}\right]\cdot\left[r\cdot e^{i\cdot\left(p\cdot dw+\frac{dw}{2}\right)}\right]^{n-1}\cdot r\cdot e^{i\cdot\left(p\cdot dw+\frac{dw}{2}\right)}\cdot\left(\frac{1}{2\cdot\pi}\right)\cdot dw$$

The causal right-sided sequence is found using the inverse z-transform with ROC outside the poles, here we choose the circular contour at radius r=10 in the z-plane for inverse z-transform:

$\text{invZ2}(H, 2, -1) = -1.244\times 10^{-11}$

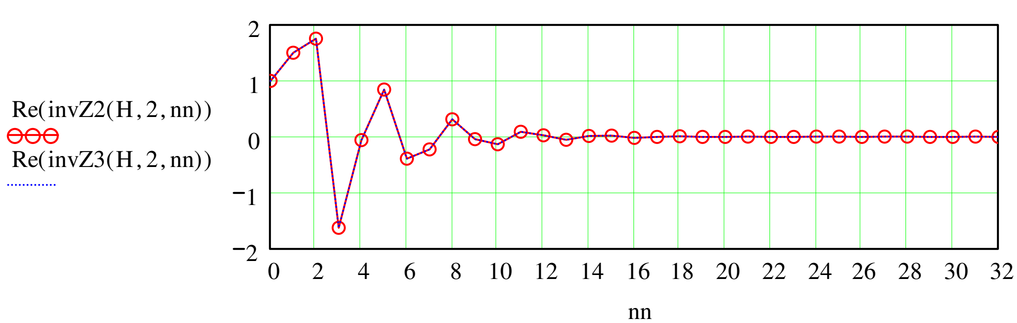
$\text{invZ2}(H, 2, 0) = 1$

$\text{invZ2}(H, 2, 1) = 1.5$

Check our approximation.

$\text{invZ3}(H, 2, 0) = 1$

$nn := 0..32$



The non-causal left-sided sequence is found using the inverse z-transform with ROC inside the poles, here we choose the circular contour at radius r=0.1 in the z-plane for inverse z-transform:

Check our approximation.

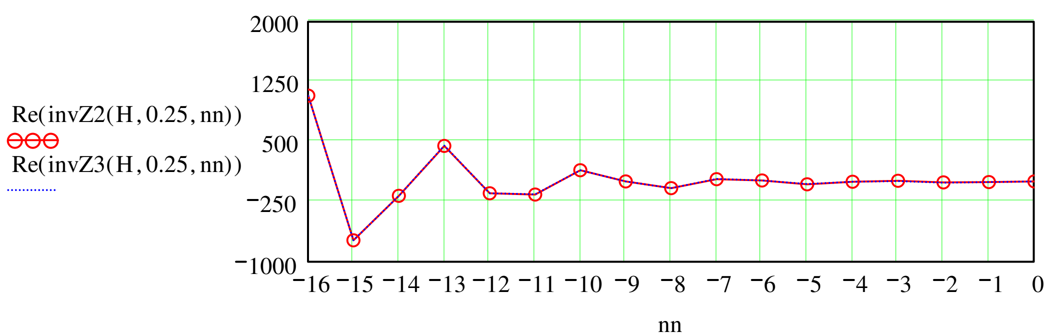
$\mathrm{invZ2}(H, 0.25, 1) = 5.1 \times 10^{-12}$

$\mathrm{invZ2}(H, 0.25, 0) = 6$

$\mathrm{invZ2}(H, 0.25, -1) = -2$

$\mathrm{invZ3}(H, 0.25, 0) = 6$

$nn := -16 .. 0$



Q11. Are the the left-sided sequence, right sided sequence, both sequences, or neither BIBO stable after redefining A(z) and B(z) in Q9 above?