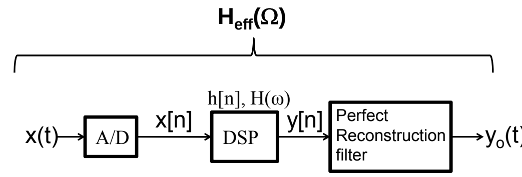
ECGR 6114
Filter Design

Ref Chapter 7 Oppenheim & Schafer 2nd Ed.

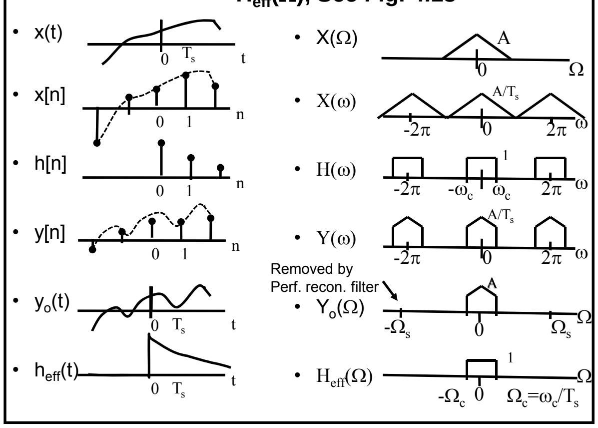
Continuous-Time Filtering with DSP

- Effective frequency response of DSP system $\mathbf{H_{eff}(\Omega)}$



- Note: "filter design" H(ω) or h[n] can be any LTI system!!
- See Figures 4.12, 4.13

$H_{eff}(\Omega)$, See Fig. 4.13

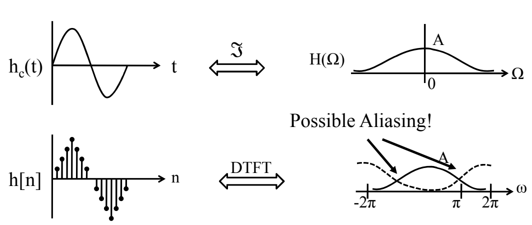


Filter Design Methods

- Generally start with some Analog Filter design, $h_c(t)$ or $H_c(s)$
- Somehow convert this to a digital filter, h[n] or H(z)
- The following methods be used to achieve this
  - Impulse Invariance Method, Section 7.1.1
  - Bilinear Transform Method, Section 7.1.2
  - Pre–warped Bilinear Transform, Section 7.1.2
  - Windowing Methods, Section 7.2

Impulse Invariance, Time Domain Method

- A. Impulse Invariance In Time Domain
  - $h[n] = T_s\, h_c(nT_s)$
  - i.e. sample the continuous impulse response at T_s



Impulse Invariance, Frequency Domain Method

- B. Impulse Invariance In Frequency Domain
  - Sometimes we want to design the Digital filter from the Laplace Transform of the Analog Filters
  - $L\{h_c(t)\} = H_c(s)$
- Derivation of the method
  - Suppose $H_c(s) = \sum_{k=1}^{N} (A_k/(s - s_k))$ (partial fraction expansion)
  - Then, $h_c(t) = L^{-1}\{H_c(s)\} = \sum_{k=1}^{N} A_k e^{s_k t}$
  - Then, $h[n] = T_s\, h_c(nT_s) = T_s \sum_{k=1}^{N} A_k e^{s_k n T_s}$

Impulse Invariance, Frequency Domain Contd.

Taking z-transform of both sides:

- $Z\{h[n]\} = Z\{T_s \sum_{k=1}^{N} A_k e^{s_k n T_s}\}$

  $= \sum_{n=-\infty}^{\infty} T_s \sum_{k=1}^{N} A_k e^{s_k n T_s} z^{-n}$

  $= \sum_{k=1}^{N} T_s A_k \sum_{n=-\infty}^{\infty} (e^{s_k T_s} z^{-1})^n$

- $H(z) = \sum_{k=1}^{N} (A_k T_s) / (1 - e^{S_k T_s} z^{-1})$

Impulse Invariance, Frequency Domain Contd.

- Given, $H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$
- Frequency domain version of Impulse Invariance method:

$$H(z) = \sum_{k=1}^{N} \frac{A_k T_s}{1 - e^{s_k T_s} z^{-1}}$$

Impulse Invariance

- Important aspects of Impulse Invariance
  - Time and Frequency domain methods are equivalent
  - Aliasing
  - Stable Analog Filter guarantees stable H(z)

Bilinear Transform

- Recall, impulse invariance has aliasing
- Bilinear Transform avoids aliasing by "Squeezing" $-\infty < \Omega < \infty$ into $-\pi < \omega < \pi$
- It is a Frequency Domain method
- Given analog filter $H_c(s)$

$$H(z) = H_c(s)\Big|_{s = \frac{2}{T_s}\frac{z-1}{z+1} = \frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}}$$

Bilinear Transform

- Important aspects of Bilinear Transform
  - No Aliasing
  - Stable Analog Filter guarantees stable H(z)
  - Frequency warping
- Since the $-\infty < \Omega < \infty$ continuous frequency is mapped into $-\pi < \omega < \pi$ discrete-time frequency, something must be "Warped" or "Squeezed"

Bilinear Transform

- Let $s = \sigma + j\Omega$; $Z = re^{j\omega}$
- Then the substitution becomes
  $s = 2(z-1) / T_s(z+1)$, which implies that
  $\sigma + j\Omega = (2/T_s)(re^{j\omega} - 1)/(re^{j\omega} + 1)$
- After much rearrangement:

$$\sigma = \frac{2}{T_s}\frac{r^2 - 1}{1 + r^2 + 2r\cos(\omega)} \qquad \Omega = \frac{2}{T_s}\frac{2r\sin(\omega)}{1 + r^2 + 2r\cos(\omega)}$$

Bilinear Transform

- Of particular interest is the unit circle in z-plane where r=1, then σ=0, and

$$\Omega = \frac{2}{T_s}\frac{\sin(\omega)}{1+\cos(\omega)} = \frac{2}{T_s}\tan(\omega/2)$$

$$\omega = 2\tan^{-1}(\Omega T_s/2)$$

- This is the "Warping" found in Bilinear Transform
- Note: for small ω only,

$$\Omega = \frac{2}{T_s}\frac{\sin(\omega/2)}{\cos(\omega/2)} \approx \frac{2}{T_s}\sin(\omega/2) \approx \frac{2}{T_s}(\omega/2) \approx \frac{\omega}{T_s}$$

Pre-Warped Bilinear Transform

- By adjusting $H_c(s)$, we can *correct* for frequency warping at "one" frequency
- Consider an N-th order Butterworth

$|H_c(\Omega)|^2 = 1/(1+(\Omega/\Omega_c)^{2N})$

- Where Ω_c is a constant – cutoff frequency in rad/s, if we designed this with Bilinear Transform, we know $\Omega = (2/T_s)\tan(\omega/2)$, so the frequency response of the Digital filter is

$$|H(\omega)|^2 = |H_c(\Omega)|^2\Big|_{\Omega=(2/T_s)\tan(\omega/2)} = \frac{1}{1+\left(\frac{(2/T_s)\tan(\omega/2)}{\Omega_c}\right)^{2N}}$$

Pre-Warped Bilinear Transform

- The cutoff frequency occurs at

  $\Omega_c=(2/T_s)\tan(\omega/2)$ or $\omega = 2\tan^{-1}(\Omega_c T_s/2)$

  when we would expected it at $\omega = \Omega_c T_s$
- So, first Pre-warp the Analog filter such that:

  $\Omega_c' = (2/T_s)\tan(\Omega_c T_s/2)$

Then

$$|H(\omega)|^2 = \frac{1}{1+\left(\frac{(2/T_s)\tan(\omega/2)}{(2/T_s)\tan(\Omega_c T_s/2)}\right)^{2N}}$$

- So:
  - First change the Analog Filter cutoff frequency so the Digital Filter cutoff is correct ($\omega = \Omega_c T_s$),
  - Then apply the bilinear transform as usual

Butterworth Filter Example

- Butterworth Filters

$$|H_c(s)|^2 = \frac{1}{1+(s/j\Omega_c)^{2N}}$$

- N = order of the filter
  - ½ power point @ $s = j\,\Omega_c$

Butterworth Poles

$$|H_c(s)|^2 = \frac{1}{1+(s/j\Omega_c)^{2N}}$$

- Poles are at :

$$(s/j\Omega c)^{2N} = -1$$

$$(s/\Omega c)^{2N} = -(j)^{2N} = -(-1)^N = \begin{cases} 1 & N\,odd \\ -1 & N\,even \end{cases}$$

- So, for N odd $\quad s/\Omega c = \sqrt[2N]{1}$

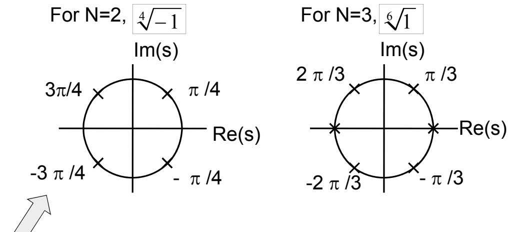
  2N roots of 1 on unit **circle**

for N even $\quad s/\Omega c = \sqrt[2N]{-1}$

2N roots of -1 on unit **circle**

Misc stuff

For N=2, $\sqrt[4]{-1}$ For N=3, $\sqrt[6]{1}$



Select the two stable poles in left-half s-plane

$$H_c(s) = \frac{1}{(s/\Omega_c - e^{j3\pi/4})(s/\Omega_c - e^{-j3\pi/4})} = \frac{\Omega_c^2}{s^2 + s\Omega_c\sqrt{2} + \Omega_c^2}$$

2nd order Butterworth

- 1.Impulse Invariance

$$H_C(s) = \frac{1}{\left(s/\Omega_C - e^{j3\pi/4}\right)\left(s/\Omega_C - e^{-j3\pi/4}\right)}$$

$$H_C(s) = \Omega_C\left(\frac{j/\sqrt{2}}{\left(s-\Omega_C e^{j3\pi/4}\right)} - \frac{j/\sqrt{2}}{\left(s-\Omega_C e^{-j3\pi/4}\right)}\right)$$

$$H_C(s) = T_S\Omega_C\left(\frac{j/\sqrt{2}}{\left(1-e^{\Omega_C T_S e^{j3\pi/4}} z^{-1}\right)} - \frac{j/\sqrt{2}}{\left(1-e^{\Omega_C T_S e^{-j3\pi/4}} z^{-1}\right)}\right)$$

Since, $\frac{A_k}{s-s_k} \Rightarrow \frac{A_k T_S}{\left(1-e^{S_k T_S} z^{-1}\right)}$

2nd order Butterworth

- 2.Bilinear

$$H(z) = \left.\frac{\Omega_c^2}{s^2 + s\Omega_c\sqrt{2} + \Omega_c^2}\right|_{s=\frac{2}{T_S}\frac{z-1}{z+1}}$$

2nd order Butterworth

3.Prewarp Bilinear

- Suppose F_s=8000 sample/s, T_s=1/8000, $\Omega_c = 2\pi\times1000$
- Then $\Omega_c' = (2/T_s)\tan(\Omega_c T_s/2) = 2\pi\times1055$

$$|H_P(s)|^2 = \frac{1}{1+\left(\frac{s}{j\frac{2}{T_s}\tan\left(\frac{\Omega_c T_s}{2}\right)}\right)^4} = \frac{1}{1+\left(\frac{s}{j(1055)2\pi}\right)^4}$$

$$H_P(s) = \left.\frac{\Omega_c^2}{s^2 + \sqrt{2}\Omega_c s + \Omega_c^2}\right|_{\Omega_c=(1055)2\Pi}$$

So,

$$H(z) = H_P(s)\Big|_{s=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right)}$$

Chebyshev Filters

$$|H(s)|^2 = \frac{1}{1+\varepsilon^2 C_n^2\left(s/j\Omega_C\right)}$$

- ε^2 sets passband ripple
- Passband varies from $1/(1+\varepsilon^2)$ to 1
- $C_n(.)$ is defined recursively

$$C_n(.) = 2xC_{n-1}(x) - C_{n-2}(x)$$

Chebyshev Filters

- Where

$$C_0(x) = 1$$

$$C_1(x) = x$$

$$C_2(x) = 2x^2 - 1$$

$$C_3(x) = 4x^3 - 3x$$

$$C_4(x) = 8x^4 - 8x^2 + 1$$

- Notes $1.C_n(1) = 1$ for all $C_n(\ )$

$2.|C_n(1)| = 1 or 0$

Freq Response from Poles & Zeroes

TEXT P.260

(2)

$$H(z) = \frac{b_0}{a_0}\frac{z^{-M}}{z^{-N}}\frac{(z-c_1)(z-c_2)\cdots(z-c_M)}{(z-d_1)(z-d_2)\cdots(z-d_N)}$$

$$= \frac{b_0}{a_0} z^{N-M}\frac{(z-c_1)(z-c_2)\cdots(z-c_M)}{(z-d_1)(z-d_2)\cdots(z-d_N)}$$

So $|H(z)| = \left|\frac{b_0}{a_0}\right| \left|z^{N-M}\right| \frac{|z-c_1||z-c_2|\cdots|z-c_M|}{|z-d_1||z-d_2|\cdots|z-d_N|}$

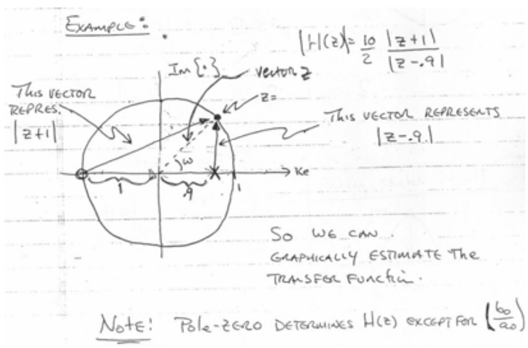
= 1 on unit circle.

Each term equals the length of a vector in z-plane.

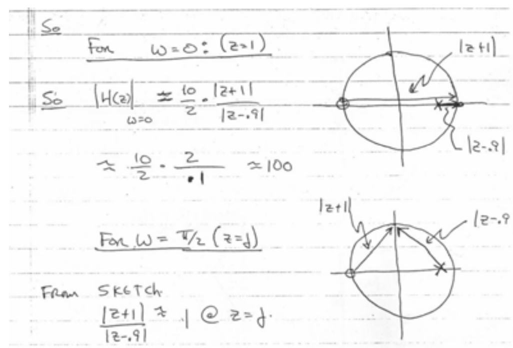
In particular:

$$|H(\omega)| = |H(z)|_{z=e^{j\omega}}$$

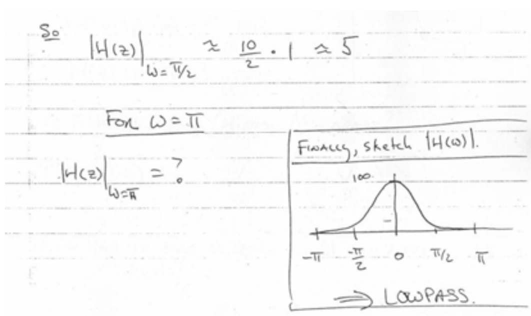
Freq Response from Poles & Zeroes



Freq Response from Poles & Zeroes

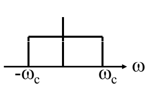


Freq Response from Poles & Zeroes



FIR Design by Windowing

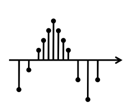
- Suppose: you want to approximate an Ideal Filter (Low Pass)



- Problem:

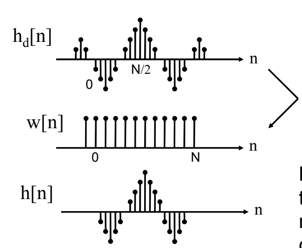
  h_d=sin(ω_cn)/(πn) is Non-Causal

  Desired impulse response



FIR Design by Windowing

- So, assume that you want M-point FIR approximation;
- And: delay h[n] by M/2 to get causal h[n]



One approach to FIR is to simply truncate, same as multiplying by w[n]

But: what is the effect of this truncation or window multiplication in frequency domain?

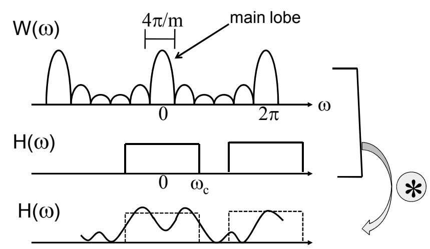
FIR Design by Windowing

- $\Im\{h_d[n]w[n]\} = \Im\{h[n]\}$
  $= (1/2\pi)\int_{-\pi}^{\pi} H_d(\alpha)W(\omega-\alpha)d\alpha$
- So, the resulting filter is the periodic convolution (Eq. 2.172) of the desired response $H_d(\omega)$ with the DTFT of the window function w[n]
- In our case $w[n] = \begin{cases} 1 & 0 \le n \le m-1 \\ 0 & \text{otherwise} \end{cases}$

So, $W(\omega) = \sin(\omega\, m/2)e^{-j\omega(m-1)/2} / \sin(\omega/2)$

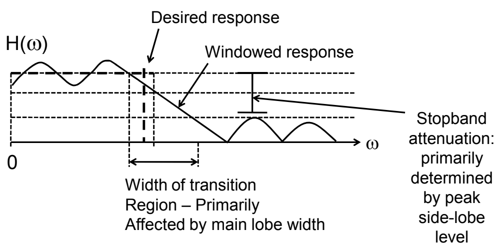
FIR Design by Windowing

- So, $W(\omega) = \sin(\omega\, m/2)e^{-j\omega(m-1)/2} / \sin(\omega/2)$

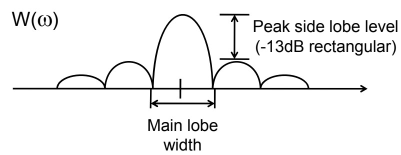


FIR Design by Windowing

- So,



FIR Design by Windowing



- So, choice of different window functions w[n] leads to different SPECTRAL tradeoffs.
  - Generally, larger mail lobe implies a lower “side lobe”
- Other windows, text book – page 468, table 7.1

FIR Design by Windowing (Sec 7.3)

- Multiply desired impulse response $h_d[n]$ by finite length window function w[n]
- Frequency domain effect:
  - $\Im\{h_d[n]w[n]\} = (1/2\pi)\int_{-\pi}^{\pi} H_d(\alpha)W(\omega-\alpha)d\alpha$
- Resulting filter is the periodic convolution of the desired response $H_d(\omega)$ with the DTFT of the window function, $W(\omega)$
- Window Functions: see Table 7.1
  - rectangular, Bartlett (triangular), etc.
  - Tradeoffs: peak sidelobe, main lobe width

Filter Architecture (Section 6.1)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k'=0}^{M} b_k \cdot z^{-k'}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Beware of sign difference
In book Fig 6.3

Generally $a_0 = 1$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k'=0}^{M} b_k z^{-k'}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{N(z)}{D(z)}$$

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] \ldots\ldots + b_0 x[n] + b_1 x[n-1] + \ldots\ldots$$

$$Y[z] = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) \ldots\ldots + b_0 X(z) + b_1 z^{-1} X(z) + \ldots\ldots$$

FIR Filter Windows

Rectangular

$$w[n] = \begin{cases} 1 ; & 0 \le n \le N_o \\ 0 ; & \text{otherwise} \end{cases}$$

Where N_o = window length

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/N_o ; & 0 \le n \le N_o/2 \\ 2-2n/N_o ; & N_o/2 \le n \le 0 \\ 0 ; & \text{otherwise} \end{cases}$$

Hanning

$$w[n] = \begin{cases} \frac{1-\cos(2\pi n/N_o)}{2} ; & 0 \le n \le N_o \\ 0 ; & \text{otherwise} \end{cases}$$

Hamming

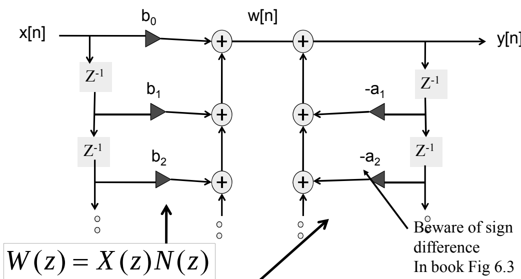
$$w(n) = \begin{cases} .54 - .46\cos\left(\frac{2\pi n}{N_o}\right) ; & 0 \le n \le N_o \\ 0 ; & \text{otherwise} \end{cases}$$

Blackman

$$w[n] = \begin{cases} .42 - .5\cos\left(\frac{2\pi n}{N_o}\right) + .08\cos\left(\frac{4\pi n}{N_o}\right) \\ 0 & \text{otherwise} \end{cases}$$

Filter	Peak Sidelobe	Mainlobe Width
Rect.	-13 dB	$\sim 4\pi/N_o$
Bartlett	-26 dB	$8\pi/N_o$
Hanning	-30 dB	$8\pi/N_o$
Hamming	-40 dB	$8\pi/N_o$
Blackman	-57 dB.	$12\pi/N_o$

Filter Architecture

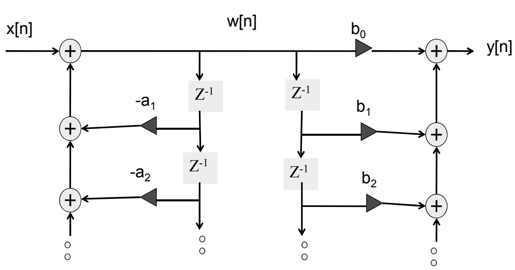


$$W(z) = X(z)N(z)$$

$$Y(z) = W(z)\frac{1}{D(z)} = \frac{X(z)N(z)}{D(z)} = X(z)H(z)$$

Filter Architecture, other forms

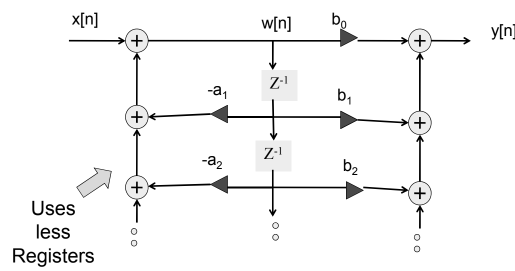
Rearrange previous page for Alternate Architecture



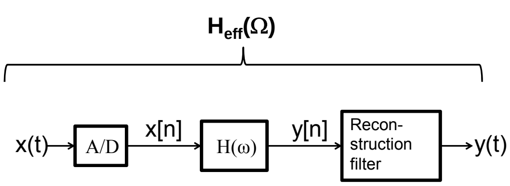
- Essentially, the above architecture interchanges the order of the N(z) and 1/D(z) sections of the filter

Filter Architecture, other forms

- The filter can be redrawn to use less registers



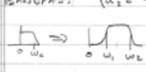
- Recall



- Note: "filter design" H(ω) or h[n] can be any LTI system!!

Filter Transformations

Type	Substitution for z^{-1}	
Lowpass to Lowpass	$\frac{z^{-1}-a}{1-az^{-1}}$	$a = \frac{\sin[(\omega_c-\omega_{cnew})/2]}{\sin[(\omega_c+\omega_{cnew})/2]}$
Lowpass to Highpass	$-\frac{z^{-1}+a}{1+az^{-1}}$	$a = \frac{\cos[(\omega_c+\omega_{cnew})/2]}{\cos[(\omega_c-\omega_{cnew})/2]}$
Lowpass to Bandpass	$-\left(\frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}\right)$	$a_1 = -2\alpha\beta/(\beta+1)$ $a_2 = (\beta-1)/(\beta+1)$



$$\alpha = \frac{\cos[(\omega_1+\omega_2)/2]}{\cos[(\omega_2-\omega_1)/2]}$$

$$\beta = \cot\left(\frac{\omega_2-\omega_1}{2}\right)\tan\left(\frac{\omega_c}{2}\right)$$

Type	Substitution for z^{-1}	
Lowpass to Bandstop	$\frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-1}-a_1z^{-1}+1}$	$a_1 = -2\alpha/(\beta+1)$ $a_2 = (1-\beta)/(1+\beta)$

Also:
Simple Lowpass to Highpass Transformation:
Multiply $h[n]$ by $(-1)^n$

Filter Transformations
(see Proakis &Manolakis)

- Type: Lowpass to Lowpass
- Substitution for Z^{-1} : $\dfrac{z^{-1}-a}{1-az^{-1}}$

$$a = \frac{\sin[(\omega_c - \omega_{cnew})/2]}{\sin[(\omega_c + \omega_{cnew})/2]}$$

Filter Transformations

- Type: Lowpass to Highpass
- Substitution for Z^{-1} : $-\dfrac{z^{-1}+a}{1+az^{-1}}$

$$a = \frac{\cos[(\omega_c + \omega_{cnew})/2]}{\cos[(\omega_c - \omega_{cnew})/2]}$$

Filter Transformations

- Type: Lowpass to Bandpass
- Substitution for Z^{-1} :

$$-\left(\frac{z^{-2}-a_1 z^{-1}+a_2}{a_2 z^{-2}-a_1 z^{-1}+1}\right)$$

$$a_1 = -2\alpha\beta/(\beta+1)$$

$$a_2 = (\beta-1)/(\beta+1)$$

Filter Transformations

- Lowpass to Bandpass

$$\alpha = \frac{\cos[(\omega_2+\omega_1)/2]}{\cos[(\omega_2-\omega_1)/2]}$$

$$\beta = \cot\left(\frac{\omega_2-\omega_1}{2}\right)\tan\left(\frac{\omega_c}{2}\right)$$

Filter Transformations

- Type: Lowpass to Bandstop
- Substitution of Z^{-1}: $\left(\dfrac{z^{-2}-a_1 z^{-1}+a_2}{a_2 z^{-2}-a_1 z^{-1}+1}\right)$

$$a_1 = -2\alpha\beta/(\beta+1)$$

$$a_2 = -(\beta-1)/(\beta+1)$$

- Also, Lowpass to Highpass transformatiom:
  multiply h[n] by $(-1)^n$

- So, recall