Solving Difference Equations

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k'=0}^{M} b_{k'}\, x[n-k']$$

$x[n] \rightarrow \boxed{\quad} \rightarrow y[n]$

Solution: LINEAR IF IC's = 0

$$y[n] = y_h[n] + y_p[n]$$

where

$y_h[n]$ = homogeneous solution (TRANSIENT, $x=0$)
$y_p[n]$ = PARTICULAR solution (FORCED, $x \neq 0$)

(1) FIND $y_h[n]$ BY GUESSING FORM of $y_h[n] = A\alpha^n$

then

$$\sum_{k=0}^{N} a_k\, y[n-k] = 0 = a_0 A\alpha^{n-0} + a_1 A\alpha^{n-1} \cdots$$

$$0 = A\left(a_0\alpha^n + a_1\alpha^{n-1} + \cdots a_N\alpha^{n-N}\right)$$
$$= A\alpha^n\left(a_0 + a_1\alpha^{-1} + a_2\alpha^{-2} + \cdots a_N\alpha^{-N}\right)$$
$$0 = A\alpha^n\alpha^{-N}\left(a_0\alpha^N + a_1\alpha^{N-1} + \cdots a_N\right)$$

— solve FOR ROOTS OF POLYNOMIAL, $\alpha_1, \alpha_2 \cdots \alpha_N$

then

$$y_h[n] = A_1\alpha_1^n + A_2\alpha_2^n + \cdots A_N\alpha_N^n$$

(Assuming simple roots)

(2) FIND $y_p[n]$ by guessing $y_p[n]$ has same general form as $x[n]$

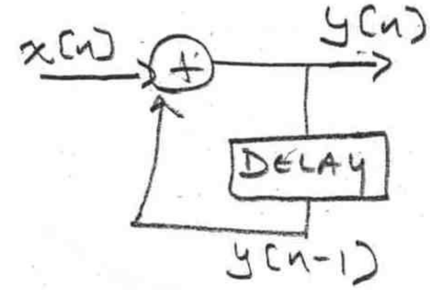
i.e., if $x[n] = \beta^n u[n]$

guess $y_p[n] = \gamma^n u[n]$.

(3) USE INITIAL CONDITIONS to solve FOR A_1, A_2, etc

Difference Equation Example

$y[n] = a\,y[n-1] + x[n]$



DIFF EQN.

$$y[n] - a\,y[n-1] = x[n]$$

Let initial conditions

$$y[0] = 2$$

And Let $a = 1/2$, $x[n] = B\beta^n u[n]$

$B = 5,\ \beta = 1/3$

(1) Homogeneous

$$y_h[n] - a\,y_h[n-1] = 0 \quad ; \text{ guess } y_h[n] = A\alpha^n$$

$$\Rightarrow A\alpha^n - Aa\alpha^{n-1} = 0$$

$$\Rightarrow A\alpha^n(1 - a\alpha^{-1}) = 0$$

$$\Rightarrow A\alpha^n\alpha^{-1}\underline{(\alpha - a)} = 0$$

$$\longrightarrow \text{ROOT} \Rightarrow \alpha_1 = a \Rightarrow y_h[n] = A_1 a^n$$

(2) Particular

$$\text{guess } y_p[n] = \text{SAME FORM} = G\gamma^n$$

$$\Rightarrow y_p[n] - a\,y_p[n-1] = x[n]$$

$$\Rightarrow G\gamma^n - aG\gamma^{n-1} = B\beta^n$$

By comparison

$$\Rightarrow G\gamma^n(1 - a\gamma^{-1}) = B\beta^n$$

Let $\gamma = \beta$

$$\Rightarrow G(1 - a\gamma^{-1}) = B$$

$$\Rightarrow G = \frac{B}{1 - a\gamma^{-1}} \Rightarrow y_p[n] = G\gamma^n$$

$\gamma = \beta.$

$$= \frac{B}{1 - a\beta^{-1}}\beta^n u[n]$$

(3) Total Solutn.

$$y[n] = y_h[n] + y_p[n]$$

$$= A_1 a^n + \frac{B}{1-a\beta^{-1}} \beta^n = A_1 \left(\tfrac{1}{2}\right)^n + \frac{5}{1-3/2} \left(\tfrac{1}{3}\right)^n$$

using initial condition

$$y[0] = 2 = A_1 \left(\tfrac{1}{2}\right)^0 - 10 \left(\tfrac{1}{3}\right)^0$$
$$= A_1 - 10$$
$$\Rightarrow A_1 = 12$$

<u>So</u> $y[n] = 12 \left(\tfrac{1}{2}\right)^n - 10 \left(\tfrac{1}{3}\right)^n$

Causal solution for causal input

$$y[n] = \left[12 \left(\tfrac{1}{2}\right)^n - 10 \left(\tfrac{1}{3}\right)^n \right] u[n]$$

<u>Check</u>